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A NEW GENUS OF BRAZILIAN CHARACIN FISHES ALLIED TO BIVIBRANCHIA

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In 1912[1] the senior author described a remarkable new Characin fish, *Bivibranchia protractila*, from the Essequibo River, British Guiana. This peculiar fish, externally remarkably similar to the marine *Albula vulpes*, was made the type of a new subfamily of the Characidæ, the Bivibranchiinæ, differing from all the others in the protractile mouth. In 1915, Steindachner reported the fish to be abundant in the Rio Branco at Boa Vista, Conceiçao, Serra Grande, and in shallow sandy reaches of the Rio Parimé. Recently Dr. Carl Ternetz has collected it in the upper Orinoco, the Rio Negro, at Santarem on the lower Amazon, and in the upper Tocantins in Goyaz, southeastern Brazil, the specimens being in Indiana University. With the material from the Tocantins are specimens of an undescribed form resembling *Bivibranchia* closely in almost all characters, but differing sharply in the very small ctenoid scales.

Atomaster, gen. nov.

Similar to *Bivibranchia* but the scales small and ctenoid, 89 to 98 in the lateral line.

Atomaster velox, sp. nov.

16181 I. Two, 160 and 190 mm., the larger the type. Riberao, Porto Nacional, Tocantins, Goyaz. Feb. 15, 1924. Dr. C. Ternetz.

16178 I. One, 129 mm. Jausinho Brook into Tocantins, Goyaz. Jan. 22, 1924. Dr. C. Ternetz.

16179 I. One, 188 mm. Santa Teresa at Barra Fazenda, Tocantins, Goyaz. Jan. 25. 1924. Dr. C. Ternetz.

16180 I. One, 122 mm. Tocantins at Peixe. Jan. 30, 1924. Dr. C. Ternetz.

16182 I. One, 104 mm. Jausinho Brook into Tocantins, Goyaz. Feb. 29, 1924. Dr. C. Ternetz.

16183 I. One, 124 mm. Carolinha, Tocantins, Maranhao. Feb. 29, 1924. Dr. C. Ternetz.

16184 I. One, 109 mm. Riberao, Lagem into Tocantins near Carolinha. March 4, 1924. Dr. C. Ternetz.

Head 4 in body length. Depth 4.8 to 5. Dorsal 11 or 12, rarely 10. Anal 10 or 11. Scales 12 to 16—89 to 98—7.

Sub-spindle shaped. Width of body equal to depth from dorsal to lateral line. Dorsal and ventral profiles nearly equally arched. Body oval in cross section. Snout narrow, projecting, rather sharp. Width of mouth 3 in length of snout. Eye 1.5 in snout, 3.5 in head, 1.1 in interorbital. Eye entirely covered by an adipose lid. Dorsal fin subtruncate, its origin equidistant from snout and middle of adipose fin. Anal small, emarginate. Caudal, deeply forked, lobes sharply pointed and about equal to head. Origin of ventrals equidistant from tip of snout and end of lateral line, the tips a little beyond halfway to origin of anal. Pectoral equal to snout and eye, reaching halfway to ventrals. Fontanel long, linear. Occipital process extending 0.2 of distance from its base to the dorsal. Anterior air-bladder about as long as eye, posterior bladder about 4 times as long, sharply conical.

Scales regularly imbricate, small on back and sides to lateral line, slightly larger below on middle of sides. Lateral line complete. Exposed portion of scale strongly convex, pectinate, the teeth graduate to the middle of the exposed part. Axillary scales of ventral long. A large adnate adipose ridge along sides above pectoral into which the fin folds.

Premaxillaries freely protractile forward, with a series of minute tricuspid teeth, as in *Bivibranchia*. Maxillary and mandible without teeth. Upper and lower pharyngeals dentigerous.

Gill membranes very narrowly united, free from isthmus. A constriction divides the gill-bearing portion of the isthmus from the scaled area behind. Gills similar to those of *Bivibranchia*, consisting of normal filaments and fleshy lobes larger than the filaments. The lobes are corrugated or divided into plates, each ridge with a recurved finger-like hook from its base. These are largest on the outer face of the upper arch of the first gill where they are branched and as high as the plate. The free portions of the lamellæ form a soft pavement. Roof of mouth with soft valve-like ridges, less conspicuous than in *Bivibranchia*.

Straw colored, each scale of the back with a group of dots. A silvery band down the middle of the side, tapering to a point anteriorly. It is above the lateral line here, but runs beneath it posteriorly.

With *Bivibranchia* this fish seems to be a plankton feeder. Both are undoubtedly very active; every line of their contours suggests speed. Their nearest relationship is probably with the Hemiodontinæ.

[NOTE: The present paper was practically the last on which Dr. Eigenmann worked. It was completed by us in 1925.—*G. S. Myers.*]

* Deceased April 24, 1927.

[1] *Freshwater Fishes of British Guiana*, Mem. Carnegie Mus., V, p. 258.

NERVOUS CONTROL OF XANTHOPHORE CHANGES IN FUNDULUS

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Communicated June 20, 1927

The killifish (Fundulus heteroclitus, Linn.), like many other fishes, is known to change in color as well as in shade to correspond in some degree with its background. Connolly ('25) found that expansion and contraction[1] of xanthophores present in the dermis are involved in these responses. There remained the question of what agency or agencies bring about the spreading or contrary movements of the yellow pigment.

Employing a method suggested by Wyman's ('24) work on the melanophores of the same species, I have ascertained the innervation of the xanthophores of Fundulus by fibers which carry impulses from the medulla, as in the case of the melanophores. If a vertical cut is made in the tail, just anteriorly to the branching of the blood vessels and in such a way as not to interfere with the circulation, a band in the tail fin distal to the cut is found to be denervated. In this band, the xanthophores after the cut is made assume almost as quickly as the melanophores the expanded condition. The change may be seen in individuals, especially females, kept over a white background and accordingly with both kinds of chromatophores in other parts of the body remaining mostly contracted. On either side of the denervated area distal to the cut the xanthophores remain contracted as long as the fish stays over the white background even if an opening has been cut to the spinal cord at about the third vertebra. But as soon as the cord so exposed is transected, there occurs an expansion of the xanthophores (as of the melanophores) in the tail and over the whole body like that observed in the narrow area of the tail fin. The previous, contracted state is not regained by the xanthophores during the two or more days which the animals so operated upon may survive in physiological salt solution. Although some blood is necessarily lost in exposing the spinal cord, the procedure entails no visible derangement of the caudal circulation. There is, therefore, good reason to conclude that the xanthophore expansion in the band distal to the cut through the posterior musculature of the tail is caused by severance of nerve fibers supplying the cells affected.

Several other facts are in harmony with the foregoing conclusion. The xanthophore expansion which occurs in the band as described is often greater than that induced normally by a yellow background. It is, furthermore, usually maintained for more than two weeks, regardless of the color of the background. But if the xanthophores contract again

(as may happen after twelve or more hours) and change in accord with changes of the background, they do so only half, or less, as fast as those in the unaffected areas. Death of the fish by sudden destruction of the central nervous system has for initial consequence expansion of xanthophores as of melanophores. When an anaesthetic, namely ether or chloretone, is applied to the gills alone, the xanthophores expand. They likewise expand somewhat if the animal is blinded. In short, von Frisch's ('12) finding of a nervous control of xanthophores in European fishes is in the main confirmed in Fundulus.

The idea has been expressed by some zoölogists that xanthophores respond inversely to melanophores. From what I have already said, it is, on the contrary, clear that the ordinary response to nerve stimulation is contraction of xanthophores just as of melanophores. Moreover, such positive response as may be seen in denervated xanthophores to high temperature and to anemia is contraction, while to ether it is expansion, responses which these non-nervous factors bring about in denervated melanophores likewise. The experiments have not revealed reciprocal innervation. Nevertheless, the two kinds of chromatophores sometimes show different behavior, for while a white background prompts contraction of both xanthophores and melanophores and a brownish-black background prompts their expansion, a yellow background indnces xanthophores to expand and melanophores to contract and a blue background the reverse.

The mechanism controlling these now parallel, now differential responses would, according to Hewer's ('26) conclusions for flatfishes, be a discriminating pigment-motor center in the central nervous system, especially the brain. Such a center is indicated in Fundulus, and regulation of the changes may normally be effected directly through the nerves to the xanthophores. But this scheme fails to account for the color "adaptation" which may occur in denervated areas; hence there must also be some other than nervous mechanism (yet activated initially through the eyes) to supersede in such cases and perhaps normally to reinforce the action of the nerves. Since the present report is preliminary, any further consideration of the non-nervous, e.g., humoral, factors may properly be left for a prospective, fuller account of my experiments, some of which are still in progress.

This work has been carried on at the Zoölogical Laboratory of Harvard University, Cambridge, Mass., and at the Woods Hole Laboratory of the U. S. Bureau of Fisheries under the supervision of Professor G. H. Parker, whose helpful counsel I am happy gratefully to acknowledge.

[1] I use the terms "expansion" and "contraction" for the sake of simplicity; imputation of amoeboid movement in the chromatophore processes is not intended.

Connolly, C. J., 1925. "Adaptive Changes in Shades and Color of Fundulus," *Biol. Bull.*, **48**, pp. 56–77.

Frisch, K. v., 1912. "Ueber die farbige Anpassung bei Fischen," *Zool. Jahrb., Abt. allg. Zool. Physiol. Tiere*, **32**, pp. 171–230.

Hewer, H. R., 1926. "Studies in Colour Changes in Fish, II-IV," *Philos. Trans. Roy. Soc. (London)*, Ser. B., **215**, pp. 177–200.

Wyman, L. C., 1924. "Blood and Nerve as Controlling Agents in the Movements of Melanophores," *J. Exp. Zool.*, **39**, pp. 73–131.

A QUANTITATIVE BASIS FOR THE RELATION BETWEEN VISUAL ACUITY AND ILLUMINATION

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1. The ability of the eye to distinguish details varies with the illumination. Measured as visual acuity—the reciprocal of the angular distance which must separate two contours in order for them to be recognized as discrete—this capacity of the eye has been investigated by numerous workers; but best perhaps by Koenig. Koenig[1] found thirty years ago that visual acuity increases very nearly as the logarithm of the intensity of illumination. His data are given in figure 1.

In spite of the familiarity of this information and the accuracy of the data, there exists no explanation for this curious but consistent dependence of visual acuity on illumination.

2. Visual acuity is an expression of the resolving power of the retina. Variations in visual acuity under comparable conditions of the dioptric mechanism, therefore, demand corresponding variations in the resolving power of the retina. The resolving power of a surface which, like the retina, is composed of sensitive elements of approximately uniform size is a function of the number of elements present per unit area. To account in such terms for the range of variation in visual acuity would suppose that the number of sensitive elements per unit retinal area can and does vary nearly a hundred-fold as the illumination changes. But the number of rods and cones in the retina is fixed anatomically. Therefore, it is necessary to assume that the number of elements in the retina can be varied functionally.

To accomplish this, let it be supposed that the sensibility of the individual rods and of the individual cones is not the same, but is distributed in relation to the intensity in the manner of populations, errors, etc., familiar from the work of statisticians. This distribution is given in

figure 2. The intensity units in the abscissas are in terms of retinal illumination; the ordinates give relative number of retinal elements specifically stimulated by the effects of a given illumination. The rod and cone curves are identical in form, but different in position and in heights of ordinates.

Figure 3 presents the integral distribution curves corresponding to the differential curves of figure 2 and, therefore, gives the *total* number of rods or cones which function at any intensity. The numerical values of the ordinates in figure 3 are arbitrarily chosen to read directly in visual acuity units, because the actual number of elements per unit area depends on the unit area chosen.

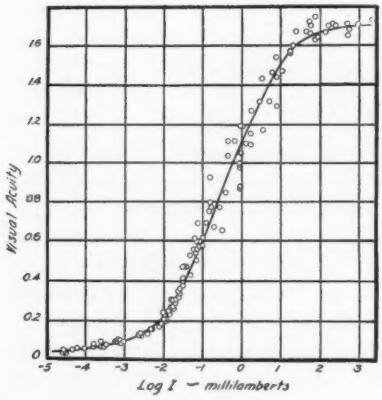

FIGURE 1

Relation between visual acuity and illumination. Koenig's original values of the intensity are multiplied by 0.072 to convert them into millilamberts.

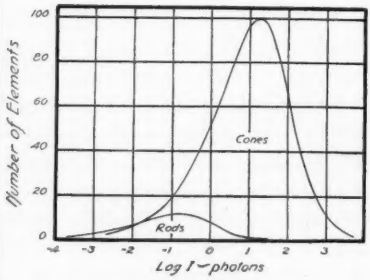

FIGURE 2

Frequency distribution curves of the thresholds of the individual rods and cones of the retina. The intensity unit represents retinal illumination and corresponds to the outside illumination in millilamberts at a standard pupil opening of 1 sq. mm

The explanation of the dependence of acuity on illumination follows from figure 3. At the lowest illuminations, vision is mediated by the rods. The number of rods which function is small, which is equivalent to a resolving surface with the receiving elements sparsely distributed. The retinal distance between two just discriminable contours must be large, and visual acuity is low. As the illumination increases, more and more rods become functional. The average distance between the functional elements becomes smaller, and visual acuity becomes larger.

Presently an intensity is reached at which the cones begin to function. Since the cones come into play nearly ten times as fast as the rods, the

number of functional cones per unit area in the fovea is soon greater than the rods in the periphery. Visual acuity then becomes a function of the foveal cones, and continues to increase steadily until all the cones are active and no further increase is possible.

The composite curve of rods and cones in figure 3 resembles the data of figure 1. In fact the curve drawn through the data of figure 1 is really this theoretical curve corrected for pupil area according to the data of Reeves.[2] The adequacy of the present explanation may thus be judged by the accuracy with which it describes the classical data of Koenig.

3. These data are for the normal eye, and the proposed explanation

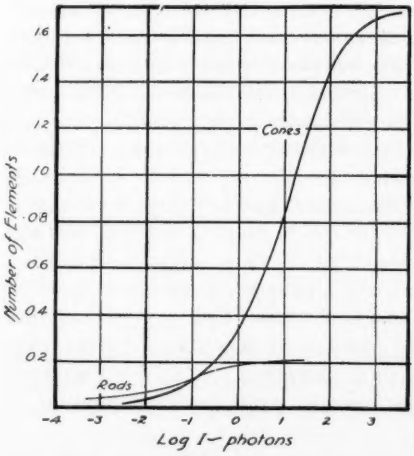

FIGURE 3

Integral frequency curves giving the total number of functioning elements per unit area at the different retinal illuminations.

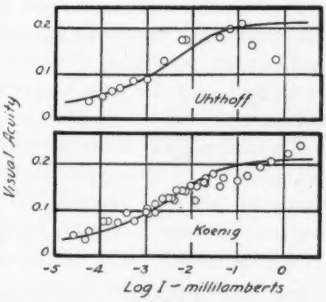

FIGURE 4

Relation between visual acuity and illumination for two completely color-blind individuals, one measured by Uhthoff and the other by Koenig. Uhthoff's original ordinates have been multiplied by 0.75 to make them comparable to Koenig's.

obviously rests on von Kries' theory of the functional separateness of rods and cones corresponding to their anatomical separation by Schultze. Koenig recognized this and attributed the lower limb of the data to the rods, and the remainder to the cones. He also recognized the implications involved, namely that in a completely color-blind individual the cone portion of the curve should disappear and leave only the lower rod limb plus any extensions of it. Figure 4 gives the data for two such individuals, one measured by Uhthoff[3] and the other by Koenig. They bear out Koenig's supposition that the two limbs of the normal eye follow von Kries' "duplicity" theory.

These data are equally significant for our explanation. Figure 3 shows that, for the normal eye, the cones overtake the rods at the middle of their range. It, therefore, follows that in a completely color-blind eye the visual acuity data should extend to a distance beyond the rod-cone intersection point equal to that which has preceded it. Moreover, the entire visual acuity data of such a case should be described by the single rod distribution curve of figure 3. The curves in figure 4 are this rod curve, the pupil correction being included as before. It is apparent that both sets of independent data are reproduced by the rod sensibility distribution curve.

4. In figure 3 the number of elements functional per unit retinal area was arbitrarily chosen to correspond directly to visual acuity units, because the actual number must depend on the size of the unit area specifically selected. It is possible, however, that there is such a thing as a minimal retinal area which carries out all the functions of the retina as a whole, such as seeing the entire range of color perception, of intensity discrimination and of visual acuity. If this were so, then the size of this minimal area, as found from the data of the different visual functions should be comparable.

From the data of intensity discrimination, Koenig computed that the whole range of intensities visible to the eye is made in only 572 discrete steps in intensity recognition. About 30 of these steps are mediated by the rods; the rest, 542, by the cones. In terms of the present notion of the sensibility distribution of rods and cones,[4] this would mean that the recognition of an intensity difference corresponds to a change from n to $n + 1$ or to $n - 1$ elements functioning in a unit area. Therefore, the minimal retinal area must contain 542 cones, or some multiple of that number.

In the same way a minimal retinal area must be able to mediate the whole range of visual acuity. The lowest visual acuity in the data of figure 1 is 0.03 unit. This corresponds to a visual angle of slightly over 44 minutes, and to a distance of 0.2 mm. on the retina. The minimal retinal area would then be 0.04 square mm. There are 13,500 cones per square mm. of fovea, from which it appears that the minimal retinal area of 0.04 sq. mm. contains 540 cones in the fovea. This is a rather striking agreement from two independent sources. A corresponding area in the periphery would then contain about 60 rods (or groups of rods, since unlike the cones, several rods are often connected to a single nerve fiber).

5. The assumption at the basis of the present explanation of visual acuity is that the rod and cone populations of the retina are like most other biological populations in the way in which their special sensory characteristic is distributed among their constituent individuals. This apparently statistical distribution of rod and cone thresholds is a funda-

mental, even if novel, point for an understanding of the visual process. Obviously, it describes the data of visual acuity. It also describes the data of intensity discrimination and of intensity recognition.[4]

In addition it can be applied to a basis for color vision. The three groups of color-sensitive fibers of Young's original color-vision theory may be considered as corresponding to three groups of color-sensitive cones. If it is then supposed that within each group of cones there exists a sensibility distribution comparable to that given here for the cones as a whole, a great many properties of color vision and of color blindness automatically follow, and are expressible in quantitative form. This supposition is here presented because it follows from the idea developed for visual acuity. The details of its application will be presented in future communications.

6. The quantitative basis here given for visual acuity is proposed independently of any mechanism to account for the statistical distribution of the sensory thresholds of the retinal elements. Nevertheless, it is necessary to point out that the form of the frequency curves of figures 2 and 3 is neither empirical nor arbitrary, but is derived from certain considerations of the photochemical properties of the retina. Studies of such visual phenomena as dark adaptation have shown that in the rods and cones there very likely exists a reversible photochemical system which may be written

$$S \underset{\text{"dark"}}{\overset{\text{light}}{\rightleftharpoons}} P + A$$

where S is a photosensitive substance, and P and A are its two principal precursors. The substances are probably different for the rods and the cones; but their dynamics may be considered as similar. Under continuous illumination by the intensity I the system comes into a stationary state described by the equation

$$KI = \frac{x^2}{a - x}$$

when a is the initial concentration of S, $a - x$ the concentration at the stationary state, x the concentration of its decomposition products P and A, and K is a constant.

In such a system the sensibility of a given retinal element depends on the concentration x of decomposition products P and A necessary to discharge an impulse to the nerve fiber. To derive the integral frequency curves of figure 3 from the equation of the stationary state it is supposed that the retinal elements are uniformly distributed in relation to the concentration x necessary to discharge an element. This amounts to making the total number of active elements a linear function of the concentration x. In relation to log I the number of functional elements

becomes identical with the stationary state curve relating x and log I, and it is this curve which is used in figure 3. Its first differential is given by the curves in figure 2.

Further details of this photosensory system are to be found elsewhere.[4] It is presented in its bare essentials here, so as to show that the frequency curves forming the basis of the present theoretical treatment of visual acuity are derived from it; and, therefore, that this phenomenon is not isolated, but explainable in terms of a mechanism which has been used for the study of other properties of vision. The complete details of this work will appear in the *Journal of General Physiology*.

[1] A. Koenig, *Gesammelte Abhandlungen zur physiologischen Optik*, Leipsic, 1903.
[2] P. Reeves, *Psychological Review*, **25**, 1918.
[3] W. Uhthoff, *Archiv Opthalmalogie*, **32** (1), 1886.
[4] S. Hecht, *J. General Physiology*, **7**, 1924.

THE FLORA OF THE HERMIT SHALE IN THE GRAND CANYON, ARIZONA

BY DAVID WHITE

U. S. GEOLOGICAL SURVEY

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The upper part, including nearly 300 feet, of the great series of red beds found in the walls of the Grand Canyon, is known as the Hermit shale. The base of this formation lies on the sandstone forming the "Esplanade," the topmost member of the Supai formation, also red, and the lowest of its beds occupy a shallow stream erosion network cut in the top of the "Esplanade." The Hermit shale consists of scattered thin sandstones and sandy shale, of a slightly warm brick red. The sediments are fresh-water laid, mainly rather fine, more or less distinctly angular sand grains, thinly coated with red oxide of iron. The shales are stream ripple-bedded and wavy. Occasional thin beds of sand evidently were rapidly deposited. Erect plants rose through and above them. Argillaceous or mud layers are largely confined to very thin slimy silt deposits that settled to form films or thin coverings on the sandy bottoms and sloping shores of quiescent pools after the run off of water.

The lower part, at least, of the formation was laid down by streams, first gradually filling the old drainage system and later building up a flood-plain over the "Esplanade" sandstone. The streams were more or less intermittent, with dry intervals in which the water in some of the shallow channels was largely or wholly evaporated. Steepness of the banks of the old waterways suggests arroyo systems cutting a great flood-plain of red sand.

Fragments of plants are found, generally rare, in the sandstones in the stream-rippled sandy shale, and in the slime layers, which sometimes also show in great distinctness footprints made by several kinds of primitive reptiles and amphibians while the slimy surfaces freshly exposed by the withdrawal of the water were still moist.

Lack of carbonaceous deposits or residues gives further evidence of intermittency of water and to conditions favorable for oxidation. The films are highly ferruginous.

The Hermit plants embrace a number of European conifers and fernlike seed-bearing plants (pteridosperms) found in Europe or closely related to European species, but about half of the flora has not been known before. A number of the forms are Uralo-Asiatic in character. The plant association is remarkable for the apparent absence of all Calamarian types and of the cosmopolitan genera *Neuropteris*, *Alethopteris* and *Pecopteris*, due to unfavorable environmental conditions such, especially, as long, dry seasons and restriction of rainfall, which are indicated also by the characters of the plants themselves as well as by the redness of the sediments. The climate may not have been much more humid than that now prevailing in northern New Mexico.

The American Paleozoic flora nearest related to the Hermit flora is that of the Wellington formation in Kansas, and the flora is probably nearly of Wellington age—i.e., Lower Permian, but not earliest Permian. It proves the Coconino sandstone which overlies the Hermit shale also to be Permian. Further, in view of the probably very short erosional break between the Hermit and the upper Supai, it becomes probable that the greater part of the latter also may be of Permian age.

LINKAGE OF SHORT EAR AND DENSITY IN THE HOUSE MOUSE

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In February, 1925, Miss Lynch very kindly sent to Dr. Castle at the Bussey Institution, a few males of the short-eared variety of the house mouse, which she had previously described. (Lynch, C. J., 1921. "Short Ears, an Autosomal Mutation in the House Mouse," *Am. Nat.*, **55**, 421–426.)

After establishing a strain of these short-eared mice, the writer decided to test the linkage relationship of the short-eared gene. Possessing a six-point recessive, namely, a non-agouti, pink-eyed, dilute, brown, piebald waltzer, it was decided to cross to this, in the hopes that possible linkage with these several characters could be tested at once.

An outcross therefore was made on this strain. Quite early in the experiment, however, it became apparent that there was linkage between density and short-ear on the one hand, and dilution and normal ear on the other, for in the F_2 generation all expected classes of young appeared except the dilute short-ear.

The character short-ear cannot be recognized in the young until they are about fourteen days old, therefore, since linkage was apparent, and since the waltzers are apt to suffer from over-crowding and loss of food in competition with normals during the latter part of the nursing period, it was decided to discard the cross to the six-point recessive and cross to a dilute strain. Dr. C. C. Little's race of dilute browns was selected and Dr. Little very kindly sent the writer several pens of these.

This cross proved to be quite satisfactory and the females were good mothers. As no strain of the double recessive, dilute-short-ear was available, the test for linkage was made by using an F_2 generation. As was to be expected, all individuals, of the F_1 generation showed both dominant characters and were normal-eared and dense.

Of the F_2 generation to date 809 individuals have been raised up to at least two weeks, or to a time when the short-eared character could be positively determined. Of these, 415 individuals have shown the characters normal ear and density, 194 normal ear and dilution, 200 short-ear and density, and none short-ear and dilution.

According to Mendelian expectation, a dihybird should give a ratio of 9:3:3:1, both dominant characters appearing in the first type, one dominant and one recessive in each of the next two types and both recessive characters in the last. It is evident, then, from the figures given above that there is not free segregation, and that a cross-over has not occurred in any two gametes which united to form one of these F_2 zygotes, which occurrence alone would produce a short-eared dilute animal.

There is the possibility that a cross-over might have occurred in one gamete only. This could not be detected in the F_2 generation, for it would not alter the phenotype. Thus, the normal-eared, dense individuals might be of any of the following formulae: $S^e s^e Dd$, $S^e S^e Dd$, $S^e s^e DD$, or $S^e S^e DD$. The second and third would indicate a cross-over in one gamete, but not in the other, while the last would indicate a crossing over in both gametes. Likewise, the short-eared, dense individuals might be $s^e s^e DD$ or $s^e s^e Dd$, and the dilute individuals $S^e S^e dd$ or $S^e s^e dd$. A genetic formula of the second-named sort (heterozygous) would in each instance indicate a cross-over in one gamete, but would not alter the appearance of the individual.

In order to test all of these possibilities, the normal-eared, dense individuals of the F_2 generation were brother-sister mated to other normal-eared, dense mice (their sibs). The mating of double heterozygotes,

S^es^eDd, would be the only one which would give all three classes in the F_3 similar to those in the F_2, and if linkage were complete, in the same ratio. Up to date 321 such individuals have been tested and all have produced the expected three classes in the ratio of 153: 83: 86, no double cross-over again being observed. Furthermore, the F_4 generation of similar matings has yielded a total of 182 in the ratio of 85: 39: 58.

Thus all matings of normal-dense individuals of the F_1, F_2 and F_3 generations have produced only the three classes with a total number of individuals of 1312, in a ratio of 653: 321: 338. Expectation under the hypothesis of complete linkage would be 652: 326: 326. The conditions of the hypothesis are quite closely met and would indicate in addition that there has not been a cross-over even in single gametes in any of the animals studied.

Similarly, as stated above, the short-eared dense and the normal-eared dilute mice of the F_2 generation might be of two possible genetic formulae each of which could be tested by mating to a dilute, and a short-ear respectively. For if the dominant factor were heterozygous in either case, indicating a cross-over in one gamete, the offspring would show the recessive allelomorph of each factor. Thus, if the short-eared dense individuals had a formula of s^es^eDd (cross-over in one gamete) instead of s^es^eDD, then mated to a dilute mouse, 50% of the offspring would be dilute. Likewise, if the dilute individuals had a formula of S^es^edd, then mated to a short-ear, 50% of the offspring would be expected to be short-eared. To date, 100 short-eared F_2, F_3, and F_4 individuals have been tested by mating to a dilute and in no case has the mating produced a short-eared mouse. Likewise, 59 dilute mice of the F_2, F_3, and F_4 generations have been tested by mating to a short-ear, and no mating has given anything but normal, dense individuals.

All three types of test matings would reveal a cross-over, either in both gametes or in one. We must, therefore, conclude that as far as these data are concerned, there is complete linkage between short-ear and density and normal ear and dilution.

There is a possibility that the double recessive, short-eared-dilute is lethal, and that this class is absent in the F_2 for this reason. The further tests of the F_2 and F_3 individuals, however, would indicate no lethal effects, for it is not apt to show in a heterozygous condition where there has been a cross-over in one gamete, and, as stated above, no such individuals have been discovered.

Summary

In a test for the linkage relationships of the character short-ear, the evidence produced indicates complete linkage between short-ear and density, equivalent to repulsion between short-ear and dilution. This

test for linkage was made by using an F_2 generation, and testing as many individuals as possible for cross-overs in single gametes in the F_3 and F_4 generations. A fuller account of this investigation will be published elsewhere.

NOTE ON CORRECTIONS TO H. A. NEWTON'S "1850 DATES" OF METEOR SHOWERS

BY WILLARD J. FISHER

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Communicated July 14, 1927

In a recent paper[1] the writer has treated the apparition dates of the Andromede (or Bielid) meteor swarms, with the purpose of searching for order in the puzzling irregularities of the rate of recession of the month and day of apparition. This was done by a method due to H. A. Newton; it consists essentially in computing a date in 1850, such that the interval between this and the historic date of apparition shall be a whole number of sidereal years. The results were displayed in a table and graphically.

Newton claims for his method an accuracy of one-tenth day, or better, so far as the computation is concerned: i.e., the errors due to the perturbations of the earth's motion would cause the longitude of the earth at the 1850 date to differ from that at the historic date by about one-tenth degree or less, both longitudes being referred to a fixed equinox.

In checking cases, to determine actual errors, it was found that in some few instances the error might be about one degree, indicating that the 1850 date might be in error by about one day.

This led to a computation of all the 1850 dates in Newton's paper of 1863[2] by the direct method, with Julian Day tables and logarithms or three-by-three multiplication tables. This is much longer than computing by Newton's formula, but is entirely free from suspicion. It turns out that there are the following errors in Newton's paper:

NO.	PAGE	TOPIC	NEWTON'S 1850 DATE	CORRECT
1	147	August shower	Aug. 5.8 to 10.8	6.8 to 11.8
2	147	November shower	Nov. 11.0 or 12.0	Oct. 30.3
3	148	March	Mar. 2.1	3.8
4	148	April	Apr. 30.8	30.5
5	148	A.D. 934, Oct. 19		

The last case above is computed by Newton as if Chinese; but the reference he gives is European. There are both Chinese and European records of showers on this date.

The second case above is altered because of a change in the accepted historic date, given in a later paper.[3] The third seems to have been

wrongly computed by Newton; his formula gives Mar. 4.8. Julian Day tables and multiplication tables give Mar. 3.8.

The uncertainty of Newton's formula affects several 1850 dates in the paper on the Andromede swarms. In consequence it is necessary to withdraw for revision all parts of that paper which depend on the 1850 dates; namely, column 3 in table 1, figure 1, and deductions from these.

As the Andromede swarms are not treated by Newton in any manner in his papers of 1863 and 1864, and as reduction to 1850 shows no advantage for the future, the reference date in the revision will be changed to 1900. References not available for the first paper will be incorporated. These changes will not much alter the average annual value of the recession of the node, but certain peculiarities will be eliminated.

[1] W. J. Fisher, *Proc. Nat. Acad. Sci.*, **12**, 1926, 728–734.

[2] H. A. Newton, *Am. J. Sci.* (2) **36**, 1863, 145–149.

[3] H. A. Newton, *Ibid.* (2) **37**, 1864, 378–381.

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*RECIPROCATING ACOUSTIC VIBRATION ON OPPOSITE SIDES OF THE PINHOLE, IN LONG QUILL TUBES**

BY CARL BARUS

DEPARTMENT OF PHYSICS, BROWN UNIVERSITY

Communicated June 23, 1927

1. *Salient Glass Pinhole with Anterior Quills.*—The sensitive glass pinhole g in figure 1 is provided with the anterior quill tube gc, of length δ, the mouth being 1 cm. from the bottom of the pipe p. The rear tube bg is long (75 cm.), connecting at the far end with the U-gauge, and s (roughly in 10^{-6} atm.) is the pressure observed there. Since bg is salient and gc reëntrant, one would expect a diminished effect in the acoustic pressure s, as compared with bg alone under like circumstances. The reverse is emphatically the case, as the nodal intensity of the highest crest is over 8 times that of the lowest trough, both obtained by anterior quill tube additions. The pipe p was 29 cm. long and all vibrations in tune.

The capacity (C) graphs of the oscillating telephone circuit had shown that the pipe p sounded loudest (C,s graphs at a sharp crest) for C = 0.12 m.f. This adjustment was, therefore, left unchanged throughout, implying fixed sound intensity. C,s relations, moreover, were examined in detail but cannot be reproduced here.

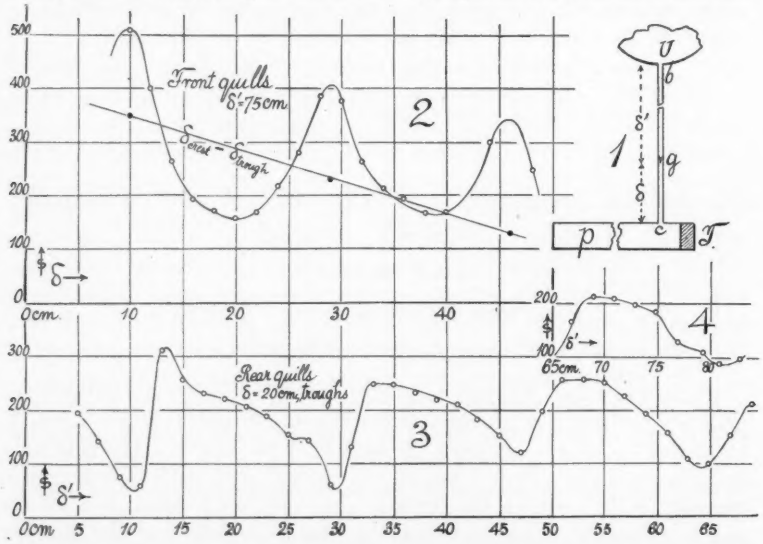
In all these experiments there are thus necessarily two pinhole probes, bg and gc, in reciprocation with each other. The efficiency of one will be

* Advance note from a report to the Carnegie Institution of Washington, D. C.

a maximum when the other is at a minimum. It was first necessary to make the survey (s,δ) of the effect of varying δ when δ' is constant and very long (75 cm.). These results are given in figure 2 as obtained with a carefully tuned pipe p. The graph is remarkably regular, with the cardinal points,

Crests:	δ	=	10	..	29	..	46	cm.
Troughs:	δ	=	..	20	..	39	..	cm.
Double amplitude:	$2a$	=	350		230		130	

so far as they can be located. The amplitude falls off slowly with δ, as usual, and the wave-length decreases. Troughs lie between the crests and



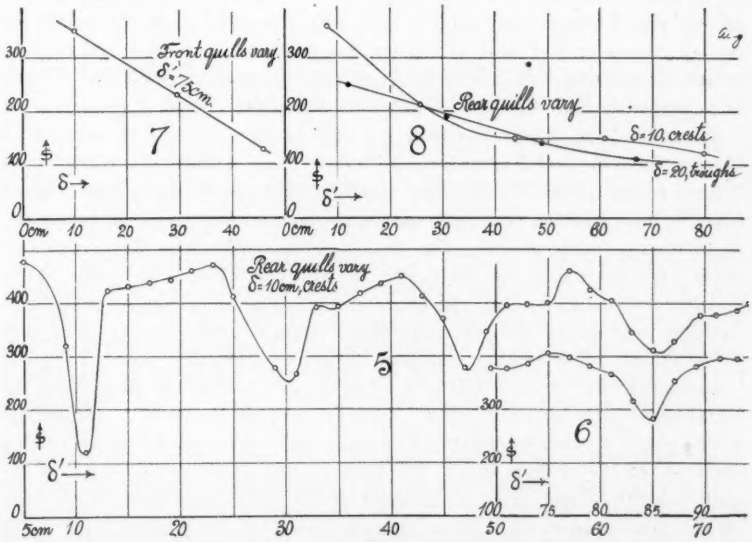
vice versa very nearly. We notice that with δ' constant, crests fall more than troughs rise, these being nearly stationary. If the successive double amplitudes ($s_{\text{crest}} - s_{\text{trough}} = 2a$) be coördinated with the mean crest position δ, the graph (Figs. 2 and 7) is nearly linear and $2a = 0$ should occur at $\delta = 68$ cm.; but the graph is more liable to become asymptotic. Within the range observed $2a = 410 - 6\delta$. The regular sinuosity of figure 2 is to be associated with the long rear quill, δ', as will presently appear.

2. *Rear Quill Tubes* (δ') *Variable.* δ *at Crest Value.*—The quill tube gc was now kept at the length $\delta = 10$ cm., corresponding to the maximum in figure 2, while bg was varied in length from $\delta' = 5$ cm. to 93 cm. The results of the survey, given in figures 5, 6 (supplementary), are quite peculiar and decisive. It is in the first place seen that crests in δ' (Fig. 5) coincide in quill tube length with troughs in δ (Fig. 2). This ceases to be fully the

case when δ is very large; but one can hardly expect that the joining together of these long lengths of quill tube can be made without some shift or extraneous resistances. Moreover, there is the cone at g which introduces uncertainty at the abutting δ,δ' lengths.

The second and third crests in figure 5 are astonishingly truncated over a wide δ' interval. The fringe displacement, s, in these experiments, moreover, was not quite steady, but fluctuating over a small interval, possibly to be associated with variation in the action of the electric spring break. Granting this, the earlier crests and troughs may be thus placed:

Troughs:	δ'	=	11	..	30	..	47	.. cm.
Mean crests:	δ'	=	..	17	..	37	..	57 cm.



where the appearance is of troughs falling midway between crests. The close resemblance of this arrangement with the preceding (δ) is striking; but now the *crests* are nearly stationary in value while the troughs rise rapidly; δ is constant.

The high acoustic pressures registered here is to be noticed. The first crest gives s = 530 cm., which corresponds to more than a third of a millimeter of mercury. That such pressures are producible by vibration in long quill tubes and can be communicated through a pinhole is quite unexpected.

In the prolongation of figures 5 to 6 the crests are again almost stationary in their s values, while the troughs are still sharp and rise rapidly with δ' to a limit. Their position is

δ' =	11	30	47	65	85 cm.

so that the wave-length here cannot be said to decrease, rather the reverse. The truncated crests which eventually tend to become sinuous cannot be similarly defined as to their δ' position. We may, however, construct the s-difference of crest and the ensuing trough and assume this to hold for their mean δ' position. Thus the amplitudes $2a = s_{\text{crest}} - s_{\text{trough}}$ are obtained.

(Mean) δ' =	8	26	44	61	80 cm.
$2a$ =	360	210	150	150	120

These values are also given in figure 8. At the outset the $2a$ for the crests graph is much in excess of the $2a$ for the following trough graph, as one might expect; but after a quill pipe length of $\delta' = 30$ cm. is exceeded, the march of both is not so very different, and they indicate an eventual oscillation of s over an amplitude somewhat below $2a = 100$. It is probably that figure 7 would take the same course if prolonged.

3. *Rear Quill Tubes Variable. δ at Trough Value.*—If instead of placing the δ quill at a crest length ($\delta = 10$ cm.), it had been placed at a trough length ($\delta = 20$ cm.), this should merely drop the graph toward the abscissa; and the troughs might even become negative if δ' is not the dominating pinhole probe. The results actually obtained (Figs. 3 and 4) show more than this; for the curve has appreciably changed form. The troughs are still sharp, but the truncated crests are less interpretable than in figure 5 although the preceding distribution of crests and troughs may be accepted. The tendency at the beginning toward a sawtooth graph is unmistakable. Crests fall and troughs rise, so that we now meet the behavior of both figures 2 and 5. Clearly the pinhole probe δ' dominates, as the graph is never in negative regions. One may regard δ' as measuring the node at the pinhole in δ. The pinhole acts as an embouchure to the quill bg while vibration in gc is forced by the pipe p, with excess pressure on the side of the musical instrument actuated by the pinhole.

The graph figure 3 was prolonged as far as $\delta' = 86$ cm., a part of which is raised in the diagram (Fig. 4). The data to be obtained from these experiments are summarized in the following table:

δ	SHARP CRESTS δ'	FRINGE DISPL. s	SHARP TROUGHS δ'	FRINGE DISPL. s	DOUBLE AMPLITUDE $2a = \Delta s$	MEAN δ'
20 (trough)	13	310	11	60	250	12
	33	250	29	60	190	31
	51	260	47	120	140	49
	70	210	64	100	110	67
	..	...	81	90	...	..

The sharp edged crests and troughs of the sawtooth waves are near together. The average distance of these doublets apart is about 18 cm.

and there is again no marked diminution of wave-length with increasing tube length, δ', here. The double amplitudes decrease as shown in figure 8 slowly and retardedly (the first three almost linearly) so as to reach an asymptote somewhat below $2a = 100$. This should be a steady fluctuation as the sawtooth shape becomes less abrupt, more smoothly sinuous (like figure 2) beyond one meter of quill-tube length. There is no evidence that the sinuous curve will eventually straighten out. It is again astonishing that these thin tubes (diam. 0.35 cm.) can sustain such prolonged waves for so great a distance, in spite of the viscosity of air. These $2a$ values, denoting pressures, are proportional to the residual energy values per cm.[3] at the nodes.

EXPLANATION OF THE INCOMPLETE POLARIZATION OF MERCURY RESONANCE RADIATION

BY WALTER A. MACNAIR* AND A. ELLETT

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Communicated June 20, 1927

The incomplete polarization of $1^1So - 2^3P_1$ of mercury in resonance radiation is known to be not in accord with the predictions of Breit[1] and of Heisenberg[2] based upon the Anomalous Zeeman levels appropriate to the transition in question. The investigations of MacNair upon the Zeeman effect of the hyper fine structure of this line suggest that the low polarization may be due to the peculiar behavior of the -0.0256 A.U. component.

Recently we have shown in two quite different ways that the reason for the incomplete polarization is a coupling between the parallel and perpendicular Zeeman components of the -0.0256 A.U. component, as though both perpendicular and parallel transitions arose from the *same* level of the *higher* state.

Analysis with a Lummer-Gehrcke plate of resonance radiation both in the absence and presence of a magnetic field has shown that whenever the theories of Breit and Heisenberg predict 100% polarization and we observe less the depolarization is due to the -0.0256 A.U. component.

Furthermore, we have been able to obtain 100% polarization of resonance radiation excited by plane polarized light from a resonance bulb source by placing the resonance bulb under observation in a magnetic field of 2000 gausses or more,[3] with the magnetic field parallel to the direction of the electric vibration of the exacting light. The reason for this is that under these circumstances only the four fine structure components showing 100% polarization are excited, the parallel Zeeman component

of -0.0256 being shifted far enough so that it no longer coincides with the corresponding component in the resonance source.[4]

* NATIONAL RESEARCH FELLOW.

[1] *Phil. Mag.*, **47**, 832.

[2] *Zs. Phys.*, **31**, 617.

[3] von Keussler (*Ann. Phys.*, **82**, No. 6) had previously shown that with a wide line source (arc) the polarization became complete in 7900 gausses.

[4] MacNair, these PROCEEDINGS, **13**, 430, 1927.

THE SPACE-DISTRIBUTION OF THE PHOTO-ELECTRONS EJECTED BY X-RAYS

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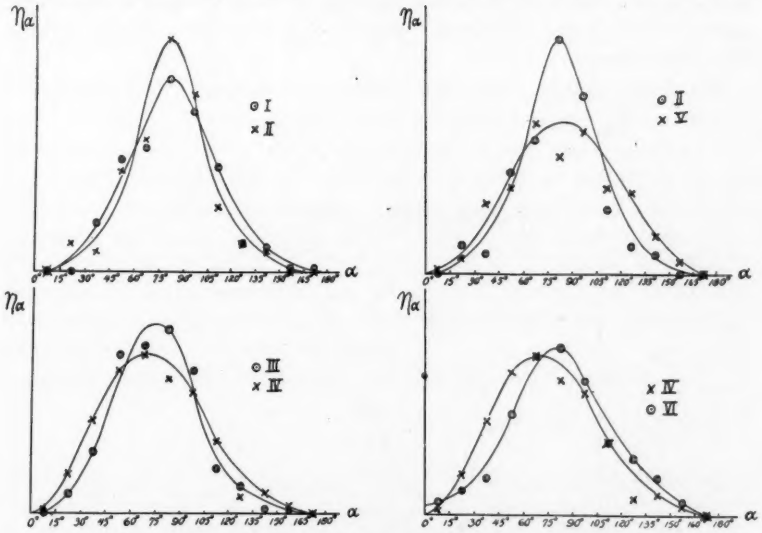
Read before the Academy April 26, 1927

The experiments of Wilson,[1] Auger,[2] Bothe,[3] Bubb,[4] Loughridge[5] and Kirchner[6] by the C. T. R. Wilson cloud expansion-chamber method have shown that while the most probable direction of the photo-electron tracks in a gas traversed by X-rays is nearly the direction of the electric vector of the incident radiation, there is apparently a very considerable "spread" in the direction of the tracks. Magnetic spectra of the electrons ejected from very thin metallic films at various angles show similar effects.[7] Theories to account for this apparent emission from the atom over a wide range of angles instead of in one definite direction have been given by Bubb,[8] Bothe,[9] Auger and Perrin,[10] Wentzel[11] and Beck,[12] but none of them is satisfactory in explaining all the now known experimental facts. The purpose of this paper is to point out that nuclear scattering of the sort postulated by Rutherford with such brilliant success in the case of α-particles has not been sufficiently considered in this connection and that it probably will account for the whole effect.

I. *Experimental Facts.*—The experiments bearing upon this question have been of two general types: (*a*) those which make use of unpolarized X-rays and study the distribution of electrons about the direction of the X-ray beam (this we shall for convenience call the *longitudinal distribution*) and (*b*) those which use polarized X-rays and study the distribution about the direction of the electric vector in a plane perpendicular to the direction of the X-ray beam (this we shall call the *lateral distribution*). It is necessary for the present paper and important, in general, that the present experimental situation be precisely stated and as this has nowhere been done, the following brief summary is given:

(1) The longitudinal distribution function has a pronounced maximum in a direction a little forward of perpendicular to the direction of the X-rays beam and falls to zero at approximately 90° either side of this maximum (this is agreed upon by all observers).

(2) The lateral distribution function has a pronounced maximum in exactly the direction of the electric vector and does *not* fall to zero at 90° from this direction (this last difference between 1 and 2 has not hitherto



I.—K-electrons of oxygen or nitrogen ejected by 15 kv. primary X-rays
II.—K-electrons of oxygen or nitrogen ejected by 20 kv. primary X-rays
III.—K-electrons of argon ejected by 80 kv. primary X-rays
IV.—L-electrons of xenon ejected by 80 kv. primary X-rays
V.—K-electrons of krypton ejected by 22 kv. primary X-rays
VI.—K-electrons of xenon ejected by 45 kv. primary X-rays

FIGURE 1

Longitudinal distribution of photo-electrons ejected by X-rays as found by Auger.

been sharply pointed out and failure to recognize it has led to some confusion).

(3) The longitudinal distribution function is asymmetrical, particularly if X-rays of high frequency are used, being steeper on one side of the maximum than on the other, while the lateral distribution function is symmetrical about the direction of the electric vector (this was first noticed by Loughridge and was communicated to the writer by him, but not published).

(4) Under similar experimental conditions the maximum of the lateral

distribution is less pronounced than that of the longitudinal or in other words, its "spread" is greater (this has not been pointed out hitherto).

(5) The amount of the "spread" decreases as the frequency of the incident X-rays increases and the atomic weight of the atoms from which the ejection takes place decreases (this is contrary to some previous conclusions).

These results are shown graphically in figures 1 and 2, the first of which represents the writer's new plotting of Auger's[13] data on the *longitudinal* distribution, the second Bubb's and Kirchner's results on the *lateral* distribution.

II. *Application of the Nuclear Theory of Scattering to the Problem.*—It

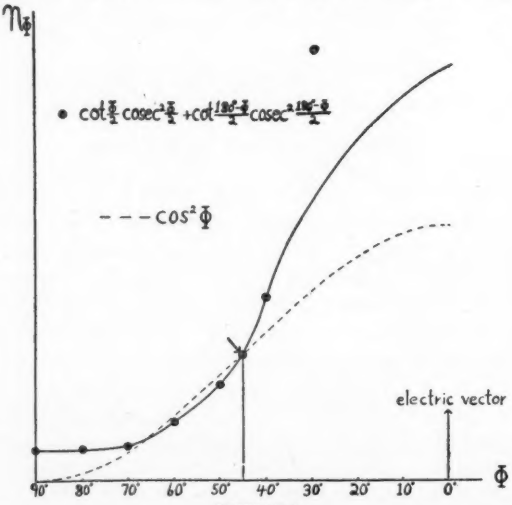


FIGURE 2

Lateral Distribution of Photo-Electrons Ejected by X-Rays as Given by the Theory of Nuclear Scattering.

has hitherto been universally assumed that the cloud expansion-chamber experiments give the actual directions of the electrons as they leave the individual atoms. This is not the case, however. The X-ray when it ejects an electron ionizes the atom and a water droplet condenses upon it. The diameter of this initial droplet was found by direct measurement on Loughridge's plates to be from 0.7 to 0.8 mm. Any change in direction of the ejected electron which took place in this distance was, therefore, unobservable. To decide then whether nuclear scattering affects the space-distribution it is only necessary to calculate the fraction of the whole number of tracks which will experience any given deflection is going a distance of 0.08 cm. If we assume that the electrons are always ejected

in the direction of the electric vector, the geometry of the situation which gives rise to the lateral distribution will be different from that which gives rise to the longitudinal distribution and we must treat, in the first case, scattering from a line, in the second case, scattering from a plane.

For the first case the nuclear theory of scattering developed by Rutherford[14] and applied by Schonland[15] to the case of electrons demands that the number dN of the electrons which are scattered between angles ϕ and $\phi + d\phi$ be given by

$$dN = \pi nNt\left(\frac{Ze^2}{2T}\right)^2\left[\cot\frac{\phi}{2}\operatorname{cosec}^2\frac{\phi}{2} + \cot\frac{180^\circ-\phi}{2}\operatorname{cosec}^2\frac{180^\circ-\phi}{2}\right]d\phi \quad (1)$$

where t is the thickness of the scattering material, n the number of nuclei per unit volume, N the total number of electrons scattered, Ze the charge on each nucleus, and T the kinetic energy of the electrons which are scattered. This equation gives not only the absolute *magnitude* of the "spread" but it also gives the *form* of the lateral distribution curve with surprising accuracy. The solid line in figure 2 was drawn so as to fit the experimental points of Bubb and Kirchner as accurately as possible. Equation (1) is represented by the circles. The dotted curve shows how inadequate in comparison is the cosine-square distribution obtained by Auger and Perrin, Wentzel and Beck.

For the case of the longitudinal distribution we get similarly

$$dN = 2\pi nNt\left(\frac{Ze^2}{2T}\right)^2\cot\phi\operatorname{cosec}^2\phi d\phi. \quad (2)$$

The distribution in this case is, therefore, a function of the whole angle instead of the half angle as in the case of the lateral distribution and so it falls to zero at 90° either side of the maximum. Moreover, since the cotangent of half the angle is always larger than the cotangent of the whole angle, the "spread" of the lateral distribution will be greater than that of the longitudinal under similar experimental conditions. The differences between the lateral and longitudinal distributions are, therefore, satisfactorily explained.

The asymmetry of the longitudinal distribution is also accounted for as follows: Since the most probable direction of ejection of the electrons is a little forward of the normal to the X-ray beam, we should have treated, instead of scattering from a plane, scattering from the surface of a cone whose axis is the direction of the X-ray beam and whose semi-angle is a little less than 90°. Such a treatment leads to the equation

$$dN = \tfrac{1}{2}\pi nNt\left(\frac{Ze^2}{2T}\right)^2\left[\frac{1}{\sin^2\frac{\alpha+\beta}{2}} + \frac{1}{\sin^2\frac{\alpha-\beta}{2}}\right]\frac{\sin\alpha d\alpha}{\cos\beta-\cos\alpha} \quad (3)$$

where α is the angle of scattering measured now from the forward direction

of the X-ray beam and β is the semi-angle of the cone. This equation gives the form of the longitudinal distribution very satisfactorily.

Equations (1) to (3) require, moreover, that the amount of the scattering and, therefore, the space-distribution depend upon both T and Z. The results of Auger given in figure 1 show this to be the case, as do also the results of other observers. Proof of this as well as of the other considerations here presented will be published in detail elsewhere.

We conclude, therefore, that the theory of nuclear scattering together with the assumption that the electrons all start from the parent atom in the same direction explains in a satisfactory way all the details of the observed space-distribution of the photo-electrons ejected by X-rays.

[1] Wilson, C. T. R., *Proc. Roy. Soc.*, **104**, 1923 (1–24).

[2] Auger, P., and C. R., *Paris Acad. Sci.*, **178**, 1924 (929–931, 1535–1536); *J. Phys. Rad.*, **8**, 1927 (85–92).

[3] Bothe, W., *Zs. Phys.*, **26**, 1924 (59–73).

[4] Bubb, F. W., *Phys. Rev.*, **23**, 1924 (137–143).

[5] Loughridge, D. H., *Ibid.*, **26**, 1925 (697–700); second paper in press.

[6] Kirchner, F., *Phys. Zs.*, **27**, 1926 (385–389; 799–801).

[7] Watson, E. C., in press.

[8] Bubb, F. W., *Phil. Mag.*, **49**, 1925 (824–838).

[9] Bothe, W., *Zs. Phys.*, **26**, 1924 (74–84).

[10] Auger, P. and Perrin, F., C. R., *Paris Acad. Sci.*, **180**, 1925 (1742–1745); *Jour. Phys. Rad.*, **8**, 1927 (93–111); Auger, C. R., *Paris Acad. Sci.*, **180**, 1925 (1939–1942).

[11] Wentzel, G., *Zs. Phys.*, **40**, 1926 (574–589).

[12] Beck, G., *Ibid.*, **41**, 1927 (443–452).

[13] Auger, P., *J. Phys. Rad.*, **8**, 1927 (85–92).

[14] See Rutherford and Chadwick, *Phil. Mag.*, **50**, 1925 (889) and references there given.

[15] Schonland, B. F. J., *Proc. Roy. Soc.*, **113**, 1926 (87–106).

SURFACE TENSION OF MOLTEN METALS. I. COPPER

BY EARL E. LIBMAN[1]

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Communicated June 27, 1927

Except for those that melt at temperatures sufficiently low to allow the use of glass vessels, no accurate data are available concerning the surface tensions of the metals. The writer is engaged in determining the capillary constants of the metals that melt above 900°C. and the present paper is an abstract of the work on copper soon to be published in detail as an Engineering Experiment Station Bulletin of the University of Illinois.

Theory.—The surface of a liquid in contact with a vertical plane which it does not wet is depressed. The magnitude of this depression h (see Fig. 1) is given by the equation

$$h^2 = a^2(1 - \sin\theta) \qquad (1)$$

where a^2 is the so-called "capillary constant" and is equal to twice the

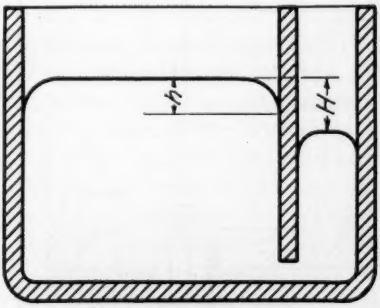

FIGURE 1

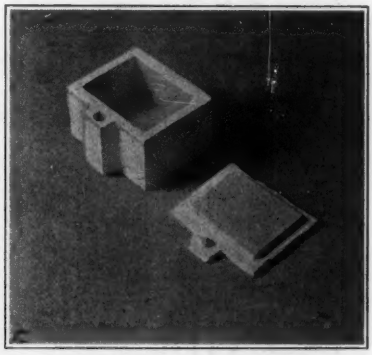

FIGURE 2

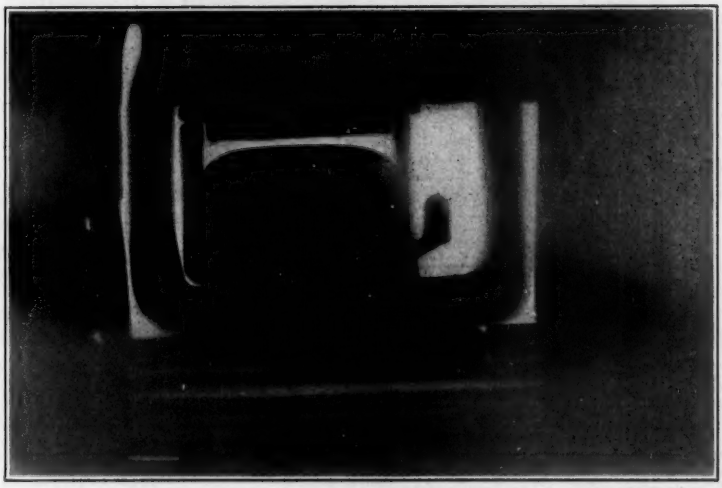

FIGURE 3

surface tension divided by the product of the density and the acceleration of gravity.

The surface of a liquid within a circular tube of radius r connected to a large reservoir is, if the liquid does not wet the tube, depressed an amount H (see Fig. 1) given by

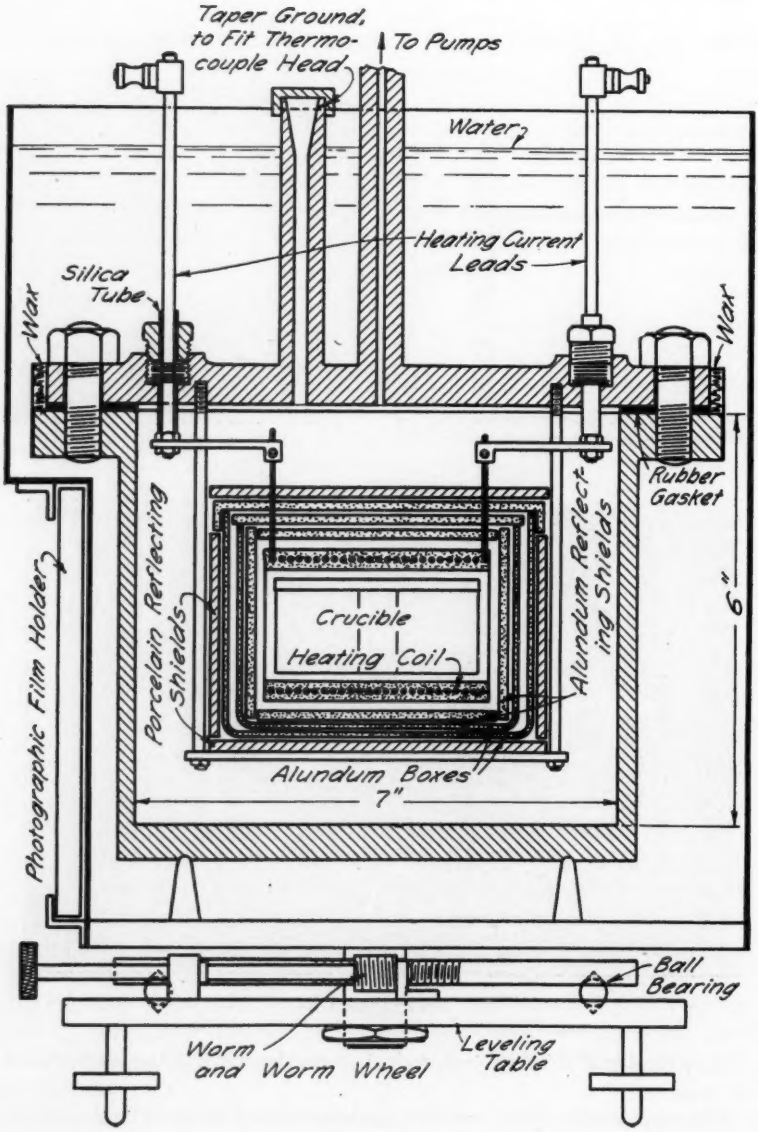
Taper Ground, to Fit Thermo-couple Head
To Pumps
Water
Heating Current Lead's
Silica Tube
Wax
Wax
Rubber Gasket
Photographic Film Holder
Porcelain Reflecting Shield's
Crucible
Heating Coil
Alundum Reflecting Shields
6"
Alundum Boxes
7"
Ball Bearing
Worm and Worm Wheel
Leveling Table

FIGURE 4

$$\left\{\begin{aligned} a^2 &= \frac{rH}{-\cos\theta} + C \\ C &= [r\alpha + \{2(\alpha^2 - m^2r^2)^{3/2}/3m^2r\} - 2\alpha^3/3m^2r]/(-\cos\theta) \\ \alpha &= a^2m^2/H \\ m^2 &= r^2/\left[\frac{a^4}{H^4} - r^2\tan^2\theta\right]. \end{aligned}\right. \quad (2)^2$$

We have thus two equations in two unknowns, a^2 and θ, and if h and H are known we may calculate a^2.

Method.—The vertical plane and the tube are combined in a single crucible shown in figures 1 and 2. The metal is melted in this crucible in a high vacuum molybdenum wound furnace. An X-ray picture is taken through the entire furnace and the photograph thus obtained (Fig. 3) is measured. From the measurements of h and H the capillary constant a^2 is calculated. The furnace is pictured in figure 4.

The furnace is operated at a vacuum which at low heats is about 10^{-7} mm. of Hg, and at high temperature (about 1400°C.) is never poorer than 10^{-3} mm. of Hg. The temperature is measured by means of the resistance of the molybdenum furnace winding. The metal at a high red heat is treated with hydrogen to obviate oxidation. During the experiments the metal continually distills, condenses upon the walls and cover of the crucible and runs down again so that the results obtained are for the metal in contact only with its own vapor.

The X-ray equipment consists of a Victor universal tube supplied by a large 20 kw. G. E. four-kenotron rectifying set. Applied to Hg, the method gave results for the surface tension ranging from 472 to 480 dynes per cm., which is in good agreement with the literature. The difficulties due to surface contamination by grease at ordinary temperatures, and to which the uncertainty in the published data for mercury is due, are not present at high temperature.

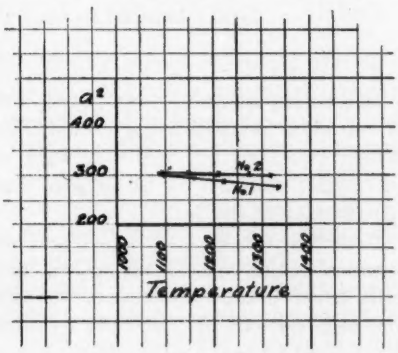

FIGURE 5

Measurements were made from the melting point to 1300°C. on two samples of copper, one of great purity and another not so pure, in order to determine the effect of slight contamination.

The pure copper was "C.P." material from Baker and Company which, after the experiments, gave on analysis a total of 0.019% of impurities, mostly Fe_2O_3 and Al_2O_3. It was originally in shot form and was melted

in an atmosphere of hydrogen to obviate oxidation. A block was cut from the center of a large cylinder thus obtained.

The impure sample, analyzed, after the experiments, gave Fe_2O_3 Al_2O_3, 0.07%, SiO_2, 0.04%. The results obtained are given below.

PURE COPPER TOTAL IMPURITIES 0.019%		IMPURE COPPER TOTAL IMPURITIES 0.11%	
	a^2		a^2
1083	0.308 ± 0.0017	1083	0.301
1093	0.308 0.0017	1097	0.299
1146	0.304 0.0017	1110	0.315 (?)
1210	0.305 0.0017	1160	0.292
1271	0.304 0.0017	1195	0.290
1318	0.297 0.0017	1220	0.287
		1335	0.275

The probable error of a^2 for the case of pure copper calculated by the method of least squares is 0.0017. Note that the effect of impurities becomes more pronounced as the temperature rises.

The results are plotted in figure 5.

[1] NATIONAL RESEARCH FELLOW.

[2] See Desains, *Ann. chim. phys.* (3) **51**, 417 (1857).

THE RELATION BETWEEN TEMPERATURE AND WORK FUNCTION IN THERMIONIC EMISSION

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Communicated July 15, 1927

In a recent article in these PROCEEDINGS[2] it was shown that some sorts of ionization accompanying chemical action not only followed the Richardson equation, but possessed such other properties in common with thermionic emission as to suggest that they might be looked upon as special cases of the latter.

Under the conditions in which ions were formed at a surface only when chemical action was taking place, distinct relationship was shown to exist between the difference in temperature of emission of the positive and negative ions (∂T) and the difference in their work functions (∂b) as determined by the usual treatment of an equation of the Richardson form.

It is the purpose of the present paper to show that an interdependence also exists between the difference in temperature (∂T) and the difference in work function (∂b) for the positive and negative thermionic currents from a given surface in the presence of various gases.

The apparatus used was similar to that described before except that a mica shield was so placed as to prevent the diffusion of air between the electrodes. The inner electrode was a thin-walled aluminum cylinder closed at one end and cooled by a current of dry air blown down the center.

A saturation current could not be obtained since the experiments were conducted at atmospheric pressure. The difference of potential between the electrodes was maintained at 225 volts. The temperature T in table 1 is arbitrarily taken as that at which the current was 1×10^{-15} amps. per sq. cm.

In table 1 are listed the data obtained from the thermionic emission from gold in the presence of various gases.

TABLE 1

GAS	CHARGE ON ION	T (ABSOLUTE)	∂T	B ($\times 10^4$)	∂B
Ethyl alcohol	+	674		2.33	
	−	705	31	2.7	0.37
Hydrogen	+	660		3.41	
	−	697	37	3.8	0.39
Oxygen	+	704		3.23	
	−	790	86	4.04	0.81
Carbon dioxide	+	722		3.35	
	−	803	81	4.2	0.85
Propyl alcohol	+	635		1.84	
	−	726	91	2.64	0.80
Helium	+	715		3.16	
	−	757	42	4.15	0.99
Butyl alcohol	+	642		2.51	
	−	753	111	3.6	1.09
Nitrogen	+	707		2.88	
	−	817	110	4.02	1.14

The data is clearly indicative of a dependence of ∂T on ∂b; where the difference in the amount of work necessary to remove a negative ion over that required to remove a positive ion is small the difference in temperature is likewise small, and vice versa. Helium offers the widest variation of any of the gases. Unfortunately, the supply of helium was limited and contaminated with traces of air so the data cannot be considered highly reliable. However, it is possible that, since helium is mon-atomic, the condition is somewhat different from the other cases.

This same interdependence between temperature and work function is shown by another series of experiments in which a changing electrode surface was used while the gas was maintained the same. A film of oxide was allowed to build up slowly on outer electrodes made of iron, copper and brass. While the surface was being oxidized consecutive determinations were made of first the positive and then the negative emission. Representative data are given in table 2.

TABLE 2

ELECTRODE MATERIAL	GAS	CHARGE ON ION	T (ABSOLUTE)	∂T	B ($\times 10^4$)	∂B
Iron	Oxygen	+	757		3.41	
		−	812	65	4.61	1.2
Iron oxide	Oxygen	+	796		3.8	
		−	797	1	3.86	0.06
Iron oxide (heavy)	Oxygen	+	793		3.92	
		−	789	4	3.64	0.28
Brass	Oxygen	+	764		3.23	
		−	810	54	5.31	2.08
Brass (oxidized)	Oxygen	+	785		3.30	
		−	760	25	3.14	0.16
Iron	Hydrogen	+	717		3.88	
		−	742	25	4.37	0.49
Iron oxide	Hydrogen	+	717		3.38	
		−	650	67	3.14	0.24
Iron oxide	Ethyl alcohol	+	733		5.3	
		−	708	25	3.7	1.6

While it is never possible to duplicate exactly surface conditions the nature and order of magnitude of the data given in the above table are always characteristic of such experiments.

The data representing various stages in the oxidation of the iron surface show clearly that the accumulation of oxide on the surface is accompanied by an increase in b and T for the positive ions which is concomitant with a corresponding decrease for the negative ions. When the work done in removing an ion is the same for both the positive and the negative the temperature is likewise the same.

The same correlation between ∂b and ∂T that was observed with the gold surface is to be noted in the emission from iron in the presence of hydrogen and of oxygen.

The emissions from iron oxide in hydrogen and in ethyl alcohol show a marked similarity in that the work necessary to remove the positive ion is abnormally high. When the alcohol vapor was mixed with an excess of oxygen to prevent a reduction of the oxide, the values of T and b obtained were those of the oxygen and not of the alcohol. It is most probable, therefore, that the small positive current obtained from alcohol or hydrogen alone on the oxide comes from the iron which has been reduced and not from the oxide. Similar results were obtained for the oxides of copper, lead and brass, but not for aluminum. Aluminum oxide, which failed to show any appreciable tendency to hold back the positive ions, is alone, of the oxides used, not capable of being reduced by hydrogen.

Without any recourse to theory or to the nature of the ions emitted, the observed interdependence between temperature and work function should serve as another method for determining contact potentials. The values of either ∂b or ∂T measure the difference in work necessary to remove

an ion of one sign over the other from an emitter in various gases, and should, therefore, be a direct measure of the change in contact difference of potential occasioned by changing the nature of the gas.

For purposes of simplicity these phenomena, characterized by an analogous behavior of the positive and negative emission currents and a definite interdependence between temperature and work function, can best be interpreted by the assumption that the gas molecules upon approaching sufficiently close to the surface are dissociated by the combined surface forces into positive and negative ions. These surface forces, dissociating molecules into ions and holding the ions thus formed to the surface, may be considered as composed of two specific forces, an electrostatic image force, and an intrinsic force. The electrostatic image force, given by $e^2/4d^2$ where e is the charge on the ion and d the distance from the surface, is appreciable to about 10^{-4} cm. from the surface, and is non-selective in character, i.e., it is the same for the positive as for the negative ion. The intrinsic force, on the other hand, is selective in character in that it allows a more ready escape of ions of one sign than of the other. For pure metal surfaces such as gold, copper and iron it will be noted that the positive ions always escape at a lower temperature than do the negative, while for the highly oxidized iron surface the conditions are just reversed. The nature of this intrinsic force, as given by its selective behavior, is that of one possessing the properties of a resultant electrical field effective to a few molecular diameters out from the surface. Such a field would be positive for iron or gold, but negative for the iron oxide.

These combined forces holding ions to the surface are opposed by the kinetic energy of agitation, so that when the temperature rises to a sufficiently high point ions will begin to escape from the surface because of their kinetic energy, and will thus come under the influence of the external applied field.

This picture accounts at once for the interdependence between temperature and work function. Just to the extent that an ion of one sign is held more tightly to the surface than is an ion of the other, to that extent is a higher temperature required for its removal, hence the existing relationship between ∂b and ∂T.

An interesting fact brought to light by this research is that it is possible to alter the nature of the surface so as to cause the intrinsic field to change sign, as is illustrated by the gradual change in b and T accompanying the accumulation of oxide on the iron surface. This change in the intrinsic field is doubtless due to the presence of the negative oxygen ions in the oxide, which, with increasing concentration on the surface, gradually neutralize the positive intrinsic field of the pure metal, and finally, upon high oxidation, actually establish a negative field.

If this mode of interpretation is correct it ought to be possible to make

direct determinations of the contact potential which must change sign as ∂T and ∂b, as observed above, pass through zero. Further experiments are in progress to test the foregoing viewpoint.

The writer's best thanks are due Professor R. A. Millikan for his interest and advice.

[1] NATIONAL RESEARCH FELLOW.

[2] These PROCEEDINGS, **12**, 560, 1926.

ON THE SPECTRUM OF ARGON IN THE EXTREME ULTRA-VIOLET

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Communicated June 30, 1927

The spectrum of Argon has recently been made the subject of several investigations which have led to the discovery of its structure. The key to success lay in finding the two strong resonance lines in the extreme ultra-violet.[1] From these as a starting point, and from ingenious experiments on the absorption of excited Argon, Meissner[2] was enabled to arrange the conspicuous group of red lines. The writer sorted out the principal series, and a list of other terms[3] and Meissner independently obtained the prin-

TABLE 1

LINES APPEARING IN THE ARGON SPECTRUM WHOSE INTENSITIES DO NOT VARY PROPORTIONATELY WITH THE TWO RESONANCE LINES

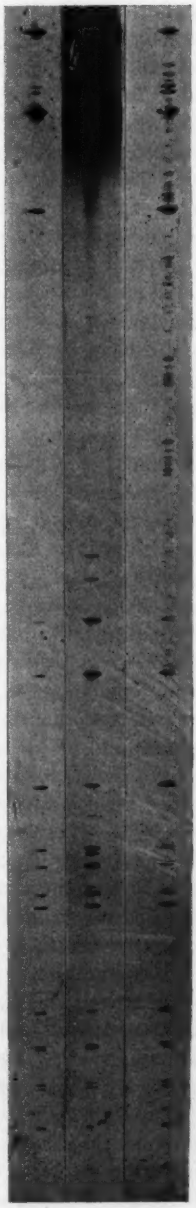
INT.	λ	MEISSNER'S ASSIGNMENT
7	894.32	$1p_0 - 3d_5$
8	879.95	$1p_0 - 2s_4$
8	876.06	$1p_0 - 3d_2$
7	869.76	$1p_0 - 2s_2$
6	866.80	$1p_0 - 3s'_1$
6	842.80	$1p_0 - 4d_5$
4	834.98	$1p_0 - 3s_4$
6	834.38	$1p_0 - 4d_2$
4	826.37	$1p_0 - 4s'_1$
4	825.33	$1p_0 - 3s_2$
2+	820.11	$1p_0 - 5d_5$
1?	816.40	
3+	816.22	$1p_0 - 5d_2$
0+	809.90	$1p_0 - 6d_5$
0+	806.65	
2+	806.44	
1+	801.22	
0+	799.21	
1	797.57	

cipal, sharp and diffuse series[4] completely. The agreement between the results of the last two researches is nearly but not quite complete. At the same time the writer included a list of the extreme ultra-violet lines of Argon which differs so markedly from the list independently measured by Dorgelo and Abbink[5] as to require explanation, which it is the main purpose of this note to supply.

The list of lines given by the writer as due to the spectrum of normal Argon in the extreme ultra-violet were those whose behavior in regard to intensities in different sources was such as to make them appear to belong physically to the same emitting system as the two resonance lines, 1048 and 1066. A prominent group of lines whose wave-lengths and intensities are given in table 1 was excluded for the reason that their intensities did not so behave. Their existence in these spectra should, perhaps, have been mentioned, as they appear to be present in all our Argon photographs.

Besides the lines given in table 1, there are others from λ 725 to 718 and from λ 679 to 662 on my lists, of the same sort. Comparison of table 1 with Meissner's list shows that nearly all these lines are classified by him as part of the spectrum of normal Argon (A I). The wave-lengths here given are in close agreemnet with those of Dorgelo and Abbink. The reason for Meissner's classification of these lines are cogent, but it seems worthwhile to record the peculiarities of their behavior which made me reluctant to regard them as part of the Argon I spectrum, and which are curious enough to call for further investigation.

The plate shows three spectra from λ 1066 to about λ 800. (*a*) is a spectrum of



Neon with Hydrogen (both atomic and molecular) and Argon as impurities. I have no way of knowing just how much Argon was present, but suppose it to have been of the order of perhaps 5 or 10%. The strength of λ 1066 and λ 1048 in (*a*) and (*b*) would indicate this amount. (*b*) is a spectrum of Argon, free from Neon, and almost free from Hydrogen. (*c*) is a spectrum of Neon, as in (*a*), but with practically no molecular Hydrogen.

In passing from (*b*) to (*a*) we see an enormous decrease in strength in the spectrum of A I as judged by the lines λ 1066 and λ 1048, while at the same time the lines λ 894 and λ 842 are increased, and others such as λ 879, 866, etc., are not very much changed. On another plate, not reproduced, the lines λ 1066 and λ 894 are of approximately equal intensity. Such facts make it hard to regard all these lines as coming from the normal Argon atom. Dorgelo and Abbink refer to this point[6] (as I had communicated some of these data to them privately) and say that these abnormal intensities may be due to the presence of Neon atoms in a metastable state. These would presumably impart energy to the Argon atoms by collisions; and such collisions would be very frequent in cases such as (*a*) or (*c*). A metastable Neon atom would be in a state indicated by the lowest possible s_3 or s_5 level and would, therefore, have on hand a store of excess energy ΔE = h.c. ν, where $\nu = 1p_0 - 1s_5$ or $1p_0 - 1s_3$, whose values are 134042 and 134819, respectively. If this amount of energy could be given by collision to the Argon atom, it would be more than enough to ionize it and produce its whole spectrum, since $1p_0$ in Argon is itself only 127109. It seems, therefore, that collisions between metastable Neon atoms and neutral Argon atoms would not explain a selective emission on the part of Argon of part of its spectrum in preference to its resonance lines.

It is plain in the reproduction that the lines λ 894, 842 and 820, which Meissner lists as successive members of the series $1p_0 - md_5$, are all enhanced in the conditions given by (*a*), even relative to the other series lines. To explain this on the basis of collisions with metastable Neon atoms we must imagine that the Argon atoms are largely prevented by these collisions from dropping into the normal state by their usual set of changes, that is, via the 2*s* states, while they pass freely through the 3*s* state. This assumption seems somewhat arbitrary.

On the other hand, I do not undervalue the reasons which make Meissner, as well as Dorgelo and Abbink, regard these lines as due to Argon. I withheld them from my Argon list because they behave like lines due to some impurity. There seems, however, to be no possible impurity to which to ascribe them, as the other likely gases would not give any lines so far out in the ultra-violet unless they were doubly or trebly ionized, and the electrical conditions were not violent enough to accomplish this. I am, therefore, now disposed to accept this whole group of lines as due to

Argon, and to leave their strange variations of intensity for further study.

The spectrum (*b*) shows four lines λλ 887, 883, 875 and 871 which were listed in my earlier article as A I. I regret that I had overlooked a measurement of these lines (with two more) by Hopfield and Dieke,[7] who have identified them without doubt as a pp' group in the spectrum A III. They are also given by Dorgelo and Abbink in the so-called "blue" spectrum of Argon, which is really to be regarded as a mixture of A II and A III, with perhaps higher spectra also represented. The presence of these lines makes it clear that the excitation which produced the spectrum (*b*) was not quite so mild as had been supposed. This particular pp' group of six lines is very prominent in the spectrum of Argon when a more violent discharge is sent through the tube, and there also then appears a group of three lines, at λλ 850.66; 843.78; 840.02, given by Dorgelo and Abbink, the first of which was recorded by me previously as A I. This may be a group in A III also, as two of its lines are separated by an amount which occurs also in the pp' group.

The spectrum (*b*) shows four interesting lines λλ 946.95, 941.39, 932.09 and 919.80 which were given in my earlier list. Of these the first two are not visible in the spectrum of a more vigorous discharge, while the others are often strong. These lines were explained as combinations between $1p_0$ and $2p_8$, $2p_6$, $2p_4$ and $2p_3$ in turn. Dorgelo and Abbink appear to regard these combinations as the result of chance, and state that λλ 932 and 919 are the strongest members of the "blue" spectrum. In the absence of more precise wave-length determinations, we must admit that the separation of this pair makes this interpretation admissible, within the limits of experimental error, if we call these lines the fundamental pair in the A II spectrum. Both sets of measurements of the pair, however (Dorgelo and Abbink's and mine), yield a value of $\Delta\nu$, different by about 8 parts in 100,000 from the value calculated from Meissner's series limits, and in precise agreement with that calculated from $2p_3$ and $2p_4$ (Meissner's $2p_1$ and $2p_3$). Also, it is to be noted that this interpretation of these lines puts them in the "forbidden" class ($\Delta j = 0$ or 2), and forbidden lines are known to occur sometimes more strongly in vigorous discharges, and thus may easily be mistaken for lines belonging to the ionized atom. It is certainly very unlikely that chance would produce agreement within the experimental error in the case of *all possible combinations* where $\Delta j = 2$ in *both* Argon and Neon; i.e., the combinations between $1p$ and $2p_4$, $2p_6$ and $2p_8$ in both cases;[8] all, that is, that could be expected to have much intensity. And if we add to this the $\Delta j = 0$ cases, of which there are now two (λ 919; $1p - 2p_3$, and λ 850 which is $1p - 3p_5$ on Meissner's notation), the probability of all this being accidental seems to be extremely small.

The plate reproduced in an earlier article on the Neon spectrum[9] shows in spectrum (*a*) a group of lines between λ 661 and λ 680, which include the

three Neon pp' combinations (the second, fourth and sixth lines in the group starting from the strong line at λ 661.9). The rest of the group is probably part of the A III spectrum, and is given by Dorgelo and Abbink.

Dorgelo and Abbink argue against the inclusion of λλ 932 and 919 in the A I spectrum because of the unusual strength of these lines in certain types of discharge, when the resonance lines become relatively weakened. If this argument is valid, it will apply with almost equal force against the inclusion of λ 894, etc., in the A I spectrum; while, on the other hand, if the latter lines must be included, on account of the way they fit into the rest of the spectrum, it seems only reasonable to consider carefully the possibility of admitting the former in spite of their odd behavior. Later on, perhaps, a comparison of the spectra of Cl I, A II and K III can be made whereby the position of the fundamental pair in the A II spectrum can be predicted. The data at present available do not appear to make this possible. If the lines at λλ 932 and 919 should then prove both to lie in the right position and to have the proper separation for this A II pair, we must accept them, but even in that case it is not impossible that λ 932 may "cover" a fainter line, nearly but not quite coincident with it, which is the combination $1p - 2p_4$, analogous to λ 662.84 of Neon.

[1] T. Lyman and F. A. Saunders, *Nature*, **116**, 358, 1925; G. Hertz and J. H. Abbink, *Die Naturwissenschaften*, **14**, 648, 1926.

[2] K. W. Meissner, *Zs. Physik*, **37**, 238, 1926.

[3] These PROCEEDINGS, **12**, 556, Sept., 1926.

[4] K. W. Meissner, *Zs. Physik*, **39**, 172, 1926, and **40**, 839, 1927.

[5] *Die Naturwissenschaften*, **14**, 755, 1926; *Zs. Physik*, **41**, 753, 1927.

[6] *Zs. Physik*, **41**, 762, 1927.

[7] J. J. Hopfield and G. H. Dieke, *Physic. Rev.*, **27**, 638, 1926.

[8] My notation for the subscripts of the p terms in Argon is like Paschen's, but unlike that of Meissner in that he ranks these terms according to numerical magnitude, whereas I prefer to arrange them according to the combination they make, i.e., their j values. Thus my $2p_4$ is Meissner's $2p_2$.

[9] T. Lyman and F. A. Saunders, these PROCEEDINGS, **12**, 92, 1926.

A REDETERMINATION OF THE NEWTONIAN CONSTANT OF GRAVITATION*

BY P. R. HEYL

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Read before the Academy April 25, 1927

The accepted value of the constant of gravitation for the past 30 years has rested upon the independent work of Boys[1] and of Braun.[2] These two experimenters obtained mean results agreeing to 4 significant figures (6.658×10^{-8} cm.^{3} g.$^{-1}$ sec.$^{-2}$) but because their separate values varied in the third figure (by as much as 4 units in Braun's work, and by 2 units in that of Boys), Poynting,[3] in his discussion of the subject, assigns the

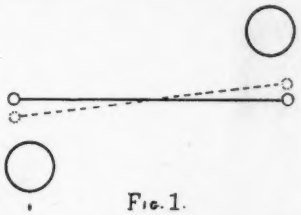

FIG. 1.

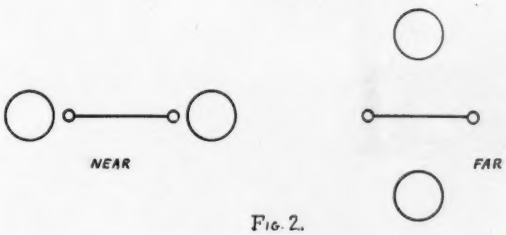

FIG. 2.

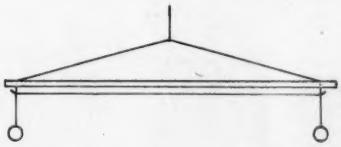

FIG. 4.

value 6.66×10^{-8}, with a possible uncertainty of one unit in the third figure.

Thirty years having elapsed since the work of Boys and Braun, it was felt that it might now be possible to push the precision of this determination one decimal place farther.

The apparatus employed was the torsion balance, used in one form or

another by half a dozen experimenters in this subject since the days of Cavendish. The general dimensions of Braun's apparatus were followed,

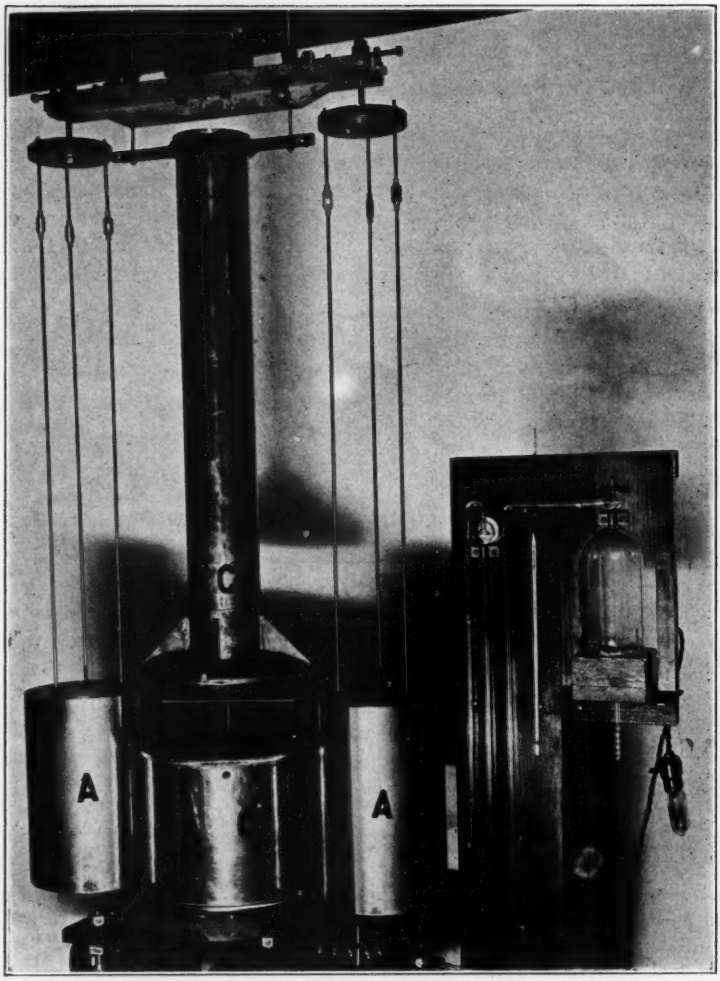

FIGURE 3

except that the large masses were greatly increased in size. The torsion pendulum was operated in a vacuum.

Two methods have been employed in measuring the constant of gravitation by means of the torsion balance. By one method, the large attracting

masses are set in the position of maximum attraction (Fig. 1), and the deviation of the pendulum from its neutral position is measured. This method was used by Boys, who pushed it so far that it seemed doubtful whether his work could be much improved on.

The second method, that of the time of swing, was used by Braun, who remarks in his paper that he had by no means exhausted its possibilities. In this method the time of swing of the pendulum is measured in two positions (Fig. 2), which may be called "near" and "far." Braun, working with large masses of about 9 kg. each and small masses of 54 grams each, with a radius of revolution of about 10 cm. obtained a difference in the two times of swing of about 46 seconds. In the present work, the large masses weighed about 66 kg. each, the other quantities being nearly the same as those employed by Braun. In consequence of the greater mass, the times of swing differed by about 330 seconds.

The present paper gives a brief description of the experimental arrangements and states the results obtained during the year 1926. An additional set of observations is being made during 1927. Publication of the complete results will be made eventually as a Bureau of Standards Scientific Paper.

The work was done in the constant temperature room of the Bureau of Standards located about 35 feet below ground. The photograph (Fig. 3) shows the general arrangement of the apparatus.

The large masses A,A, were of steel, cylindrical in shape. This form was adopted because of practical difficulties in shaping spheres of the weight employed (66 kg. each). The cylinders were turned from ingots which had been forged down from 12″ to 9″ in diameter to close up blow holes, and as finished were about 28.5 cm. long and 19.5 cm. in diameter.

As a check on uniformity of density, samples were taken from the bosses cut off from the ends of each cylinder. The density of each sample agreed with the calculated density of the corresponding cylinder to 1 part in 4000. Calculation shows that the error introduced in the result by neglecting a non-uniformity of density of this order is less than 1 part in 50,000.

The cylinders were hung by steel rods from a beam B, movable about a vertical axis, and supported ultimately by two I-beams let into the walls of the room. This axis carried at its upper end a divided circle and a vernier, reading to 0.1°.

The bell jar, C, was of metal, resting on a plate glass base. The lower wide portion was made of a piece of wrought-iron tubing for magnetic shielding; the upper narrow portion of brass. Windows for observation were provided at a number of strategical points, two of which are seen in the photograph.

The moving system may be seen hanging below the elevated upper

portion of the bell jar. It consisted of a light aluminum rod (Fig. 4) provided with a copper truss wire which passed through holes near the end of the rod, and was bent into hooks at its ends. The small masses were provided with hooks and tied together by a piece of tungsten lamp filament 0.025 mm. in diameter. This filament was laid over the hooks as shown in the figure. By this construction over 99% of the moment of inertia of the system was located in the balls.

The moving system was supported by a filament of tungsten of the same diameter and about a meter long. Tungsten has the advantages over silica of being readily tied to its connections and of being not readily broken. Boys used suspensions of silica, and says that he lost much time and previous observational labor by the frequent breaking of his suspensions for no apparent reason. It was found that with a tungsten suspension the moving system came back accurately to its resting point after large displacements.

The small masses used in 1926 were of platinum, weighing about 50 grams each, fused in a vacuum furnace to insure absence of cavities. They were spherical in form. When the upper part of the bell jar was lowered in place the balls hung nearly opposite the centers of the cylinders. The centers of the balls were about 20 cm. apart.

The usual pressure maintained in the bell jar was from 1 to 2 mm.

To start the pendulum swinging gravitational attraction was employed. Two bottles holding about 2 kg. of mercury in each were placed in the maximum attraction positions and changed to the opposite sides of the pendulum in time with its time of swing. By repeating this process for two hours it was possible to produce an amplitude of $3^1/_2$ degrees.

The time of a complete cycle swing in the near position was about 29 minutes, and in the far position $34^1/_2$ minutes. The time of swing was observed by introducing a beam of light through the glass base of the bell jar to a mirror on the moving system, and out by the same path. The motion of the reflected beam was observed by a telescope at a distance of $3^1/_2$ meters.

Successive transits of the beam (usually 24 in number) were recorded on a chronograph simultaneously with second signals from a Riefler clock.

Length measurements requiring precision, such as the distances between centers of balls or of cylinders, were measured by an optical compass similar to that used by Boys, consisting of two micrometer microscopes mounted on a rigid bar. The microscopes were sighted upon the two points whose distance was to be determined, and then upon a specially graduated standard scale. Such measurements could be repeated with a precision of 0.01 mm. Less important measurements, such as vertical differences of elevation, were made by a cathetometer reading to 0.1 mm.

The calculations involved were laborious, but not prohibitively so.

The attraction of a finite cylinder on any external point can be expressed to a precision of 1 part in 100,000 by seven terms of a zonal harmonic series.

The observations during 1926 gave five values of the constant of gravitation as follows:

	6.661
	6.661
	6.667
	6.667
	6.664
mean	6.664×10^{-8}
Average departure from mean	0.002

No previous investigator has obtained results agreeing to more than 2 significant figures.

It is planned to carry out an additional series of observations during 1927, by which it is hoped to obtain six or seven more values of this important constant.

* Published by permission of the Director of the National Bureau of Standards of the U. S. Department of Commerce.

[1] C. V. Boys, *Phil. Trans. Roy. Soc. A.*, **1895**, Part 1, p. 1.

[2] Carl Braun, *Denkschriften k. Akad. Wissens. (math. und naturwiss. Classe)*, **64**, 1897, p. 187.

[3] Article "Gravitation," *Encyclopedia Britannica*, XI Edition.

CONTACT TRANSFORMATIONS OF THREE-SPACE WHICH CONVERT A SYSTEM OF PATHS INTO A SYSTEM OF PATHS[1]

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Introduction.—A contact transformation of the surface elements of three-dimensional space[3] converts every point, curve or surface into a point, curve or surface, but not necessarily a point into a point, a curve into a curve or a surface into a surface.

An important special contact transformation is the duality. This is characterized among proper contact transformations, i.e., those which do not reduce to point transformations by the property of converting each of the straight lines of space again into a straight line.

The ∞^4 straight lines of space are an example of a system of paths. By this we mean a system of analytic curves analytically distributed[4] over a region of space so that there is one and only one curve joining any two

given points of the region, and one and only one through any given point in any given direction.

These considerations suggest the problem: to determine all proper contact transformations of space which convert the curves of some (unspecified) system of paths again into curves. This problem is solved by the following theorem, which it is the purpose of the present paper to establish.

THEOREM.—*If a proper contact transformation* Γ *of space converts the curves of a system of paths* F *into curves, then* Γ *must have the form* PDP_1 *where* D *is a duality and* P *and* P_1 *are point transformations; and* F *must be linear, i.e., convertible by a point transformation* (P) *into the straight lines.*

Proof.—Let Γ proceed from the space S containing F to the space S' containing the system of paths F' into which F is transformed by Γ.

There are three possibilities to consider: Γ converts an arbitrary point of S either into a point, a curve or a surface.

The first of these is ruled out by the hypothesis that Γ is a proper contact transformation.

If Γ converts an arbitrary point p of S into a curve C_p in S', then as p describes any curve of F, C'_p must move so that its envelope shall be a curve of F'. But this can only be when C'_p remains fixed as p describes the curve of F. It follows that C'_p must be the same for all points p of S; therefore, every union in S is carried by Γ into C'_p—a degenerate case which we may dismiss.

There remains only the case where an arbitrary point p of S is converted into a surface Σ'_p in S'. The surfaces Σ' form a triply infinite system corresponding to the ∞^3 points of S.

Imagine that p describes a curve C of F. Then, by hypothesis, the corresponding ∞^1 surfaces Σ' must intersect in the same curve C' of F', in the manner of a pencil of planes. Thus the system of paths F' consists of the mutual intersections of the ∞^3 surfaces Σ'.

Now there is a theorem of Felix Klein[5] which applies in this situation, being essentially to the effect that a system of paths F' in space which is identifiable with the mutual intersections of ∞^3 surfaces must be a linear system; i.e., there must exist a point transformation P' which converts F' into the straight lines of a projective space T'.

By applying the above reasoning to Γ^{-1}, it follows that the system of paths F can be converted by a point transformation P into the straight lines of a projective space T.

Consider the transformation $P^{-1}\Gamma P'$. Being a composition of contact transformations, it is itself a contact transformation, which proceeds from the space T to the space T'. By following the three component transformations from T to S to S' to T', we see that $P^{-1}\Gamma P'$ converts the straight lines of T into the straight lines of T'. Hence $P^{-1}\Gamma P'$ must be either a

collineation C or a duality D. It cannot be a collineation, for then Γ would be PCP'^{-1}, or a point transformation, contrary to hypothesis. Therefore, we must have

$$P^{-1}\Gamma P' = D,$$

whence

$$\Gamma = PDP'^{-1} = PDP_1,$$

where P_1 as well as P is a point transformation. This completes what was to be proved.

[1] Presented to the American Mathematical Society, May 7, 1927.

[2] NATIONAL RESEARCH FELLOW IN MATHEMATICS.

[3] Hereafter the word "space" will mean three-dimensional space.

[4] This means that in the finite equations of the paths, $x^i = f^i(t, a_1, a_2, a_3, a_4)$, $i = 1, 2, 3$, the functions f^i are analytic in the parameters a which vary from curve to curve as well as in the parameter t which varies along each curve.

[5] *Über einen Satz aus der Analysis Situs, Gesammelte Mathematische Abhandlungen, erster Band,* pp. 306–310; also *Über die sogenannte nicht-euklidische Geometrie,* in the same volume, pp. 331–343, especially pp. 333–343.

MOTION AND COLLINEATIONS IN GENERAL SPACE

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1. By a general metric space we shall understand an n-dimensional continuum in which the element of arc of a curve is given by

$$\left(\frac{ds}{dt}\right)^2 = F(x, \dot{x}), \tag{1.1}$$

where $\dot{x}$ stands for dx/dt and F is homogeneous of the second degree in $\dot{x}$.

The geometry of spaces of this sort has been studied by Finsler,[1] Synge,[2] Taylor[3] and Berwald[4] and it is the tensor analysis as developed by Berwald that we employ in the present paper. We shall denote the covariant derivative of a tensor by means of a subscript preceded by a comma, while a subscript preceded by a period will denote partial differentiation with respect to $\dot{x}$. It is understood that the summation convention of a repeated index is used.

As most of the functions with which we shall have to deal are homogeneous of degree zero in $\dot{x}$ and since we cannot equate coefficients of $\dot{x}$ as we do in Riemannian geometry or in the corresponding geometry of paths, we shall make use of the following lemma:

If $f_{i_1 i_2 \ldots i_\gamma}(x, \dot{x})\dot{x}^{i_1}\dot{x}^{i_2}\ldots\dot{x}^{i_\gamma} = 0$, *the functions* f *being homogeneous of degree*

zero in $\dot{x}$, symmetric in all subscripts and satisfying the condilions, $f_{i_1 \ldots i_\gamma, j} = f_{i_1 \ldots i_{\alpha-1} j i_{\alpha+1} \ldots i_\gamma, i_\alpha}$, then $f_{i_1 \ldots i_\gamma} = 0$.

2. By a motion in a general metric space we shall understand an extended point transformation which preserves the arc of any curve. In a coördinate system $\bar{x}$, the metric may be written as

$$\left(\frac{d\bar{s}}{dt}\right)^2 = \bar{g}_{ij}\,\dot{\bar{x}}^i\,\dot{\bar{x}}^j, \tag{2.1}$$

where $\bar{g}_{ij}$ are functions of $\bar{x}$ and $\dot{\bar{x}}$ homogeneous of degree zero in $\dot{\bar{x}}$ and symmetric in the subscripts. We apply the infinitesimal point transformation

$$\bar{x}^i = x^i + \zeta^i \delta u, \tag{2.2}$$

where ζ^i is a vector of position only and δu an arbitrary infinitesimal. If we neglect powers of δu higher than the first and make use of the lemma in §1 we find

$$g_{ih}\zeta^h_{,j} + g_{hj}\zeta^h_{,i} + \zeta^h g_{ij,h} + \zeta^h_{,l}\dot{x}^l g_{ij.h} = 0 \tag{2.3}$$

as the necessary and sufficient conditions that must be satisfied if the arc is to be preserved. We shall refer to (2.3) as the equations of Killing since they are generalizations of the corresponding equations for Riemannian geometry.[5] A rather long but not difficult calculation gives

$$\zeta^i_{,j,k} + \zeta^l K^i_{jkl} + \zeta^l_{,h}\dot{x}\Gamma^i_{jkl} = 0 \tag{2.4}$$

as the equations by means of which we can express the conditions of integrability of Killing's equations, K^i_{jkl} being the curvature tensor of the space[4] and $\Gamma^i_{jkl} \equiv \Gamma^i_{jk.l}$ where Γ^i_{jk} are the coefficients of affine connection of the space and like the g's are homogeneous of degree zero in $\dot{x}$.

Covariant differentiation of (2.4) and elimination of second covariant derivatives by means of Ricci's identities and (2.4) gives

$$\zeta^h K^i_{jkl,h} + \zeta^h_{,m}(\dot{x}^m K^i_{jkl,h} - \delta^i_h K^m_{jkl} + \delta^m_j K^i_{hkl} + \delta^m_k K^i_{jhl} + \delta^m_l K^i_{jkh}) = 0 \tag{2.5}$$

and partial differentiation with respect to $\dot{x}$ and the use of the identity

$$\Gamma^i_{jkl,m} - \Gamma^i_{jkm,l} = K^i_{klm,j} = K^i_{jlm,k} \tag{2.6}$$

gives

$$\zeta^h \Gamma^i_{jkl,h} + \zeta^h_{,m}(\dot{x}^m \Gamma^i_{jkl.h} - \delta^i_h \Gamma^m_{jkl} + \delta^m_j \Gamma^i_{hkl} + \delta^m_k \Gamma^i_{jhl} + \delta^m_l \Gamma^i_{jkh}) = 0. \tag{2.7}$$

Similar processes applied to (2.3) give by means of (2.4) and Ricci's lemma[4]

$$g_{ij,h}\zeta^h_{,k} + g_{jk,h}\zeta^h_{,i} + g_{ki,h}\zeta^h_{,j} + \zeta^h g_{ij,k,h} + \zeta^h_{,l}\dot{x}^l g_{ij.k.h} = 0 \tag{2.8}$$

and

$$g_{ij.h}\zeta^h_{,k} + g_{jk.h}\zeta^h_{,i} + g_{ki.h}\zeta^h_{,j} + \zeta^h g_{ij.k,h} + \zeta^h_{,l}\dot{x}^l g_{ij.k.h} = 0. \tag{2.9}$$

Equations (2.5), (2.7), (2.8) and (2.9) and those obtained from them by covariant and partial differentiation with respect to $\dot{x}$, the second covariant derivatives being removed by means of (2.4), must be compatible for the determination of the $n^2 + n$ functions $\zeta^h_{,i}$ and ζ^h ($\zeta^h_{,i} \equiv 0$) subject to the $(n^2 + n)/2$ conditions (2.3).

The equations obtainable from (2.5) and (2.7) by differentiation and elimination are not independent as we can show by means of (2.6); in fact if we denote the left-hand side of (2.5) by A^i_{jkl} and that of (2.7) by C^i_{jkl} we shall find that

$$C^i_{jkl,m} - C^i_{jkm,l} = A^i_{klm.j}.$$

Neither are they all equivalent as can be readily seen when we notice that in the case of Riemannian geometry (2.7), (2.8) and (2.9) vanish identically while (2.5) reduce to the corresponding equations for a Riemann space. From the preceding results we have:

The greatest number of linearly independent infinitesimal motions that a general metric space may admit is $n(n + 1)/2$.

By taking an arbitrary vector of position only, ζ^i, and normalizing it so that its components become, say, δ^i_1, we can show by means of Killing's equations that if this vector is to define an infinitesimal motion, we must have $\partial g^{ij}/\partial x^1 = 0$. Hence, we have:

The most general metric space admitting an infinitesimal motion can be obtained by choosing for the fundamental tensor a set of symmetric functions of $x^2, \ldots, x^n$ and of $\dot{x}^1, \ldots, \dot{x}^n$, the functions being homogeneous of degree zero in $\dot{x}$.

We leave the discussion of motion in flat spaces, Minkowski spaces and spaces of constant scalar curvature to a future date and go on to the problem of collineations.[6]

3. By a collineation we shall understand a point transformation which carries geodesics into geodesics or if the space be not metric but merely affinely connected, one which carries paths into paths. If the affine parameter is preserved the collineation will be said to be *affine*, otherwise *projective*. In particular, in the case of a metric space, if the arc is preserved the collineation becomes a motion.

As in §2 we begin with infinitesimal affine collineations; we consider the effect of the transformation (2.2) upon the equations of the paths in terms of an affine parameter t, namely

$$\ddot{\bar{x}}^i + \bar{\Gamma}^i_{jk}\dot{\bar{x}}^j\dot{\bar{x}}^k = 0. \tag{3.1}$$

By means of the lemma of §1 we obtain

$$\frac{\partial^2\zeta^i}{\partial x^j \partial x^k} - \Gamma^h_{jk}\frac{\partial\zeta^i}{\partial x^h} + \Gamma^i_{jh}\frac{\partial\zeta^h}{\partial x^k} + \Gamma^i_{hk}\frac{\partial\zeta^h}{\partial x^j} + \zeta^h\frac{\partial\Gamma^i_{jk}}{\partial x^h} + \Gamma^i_{jkl}\frac{\partial\zeta^l}{\partial x^h}\dot{x}^h = 0 \tag{3.2}$$

as the set of equations that ζ^i must satisfy in order to define an infinitesimal affine collineation. When (3.2) are written in tensor form they reduce to (2.4). Hence the problem of existence reduces to the one considered in §2 without imposing the conditions (2.3) and those arising from them.

The greatest possible number of linearly independent affine collineations that any space may admit is $n^2 + n$; this will be the case when, and only when, (2.5) and (2.7) are satisfied identically. This gives $K^i_{jkl} = 0$ and $\Gamma^i_{jkl} = 0$; hence, we have:

When an affinely connected space admits $n^2 + n$ linearly independent affine collineations it is necessarily metric and its coefficients of affine connection are functions of position alone.

Ricci's lemma in this case becomes $g_{ij,k} = 0$ which is a sufficient condition to insure the fact that when an arbitrary vector is carried by parallel displacement around an infinitesimal circuit, its length is unchanged (up to the third order of infinitesimals).

4. When we do not require that the affine parameter of the paths (or geodesics) be preserved under the transformations (2.2), we take the equations of the paths in the form

$$\dot{\bar{x}}^i(\ddot{\bar{x}}^j + \bar{\Gamma}^j_{kl}\dot{\bar{x}}^k\dot{\bar{x}}^l) - \dot{\bar{x}}^j(\ddot{\bar{x}}^i + \bar{\Gamma}^i_{kl}\dot{\bar{x}}^k\dot{\bar{x}}^l) = 0 \tag{4.1}$$

which are invariant in form not only under arbitrary transformations of coördinates but also under arbitrary transformations of parameter.[7] By means of the lemma of §1, the conditions on the vector ζ^i take the form

$$\zeta^i_{,j,k} + \zeta^l K^i_{jkl} + \zeta^l_{,h}\dot{x}^h\Gamma^i_{jkl} = \delta^i_j\varphi_{.k} + \delta^i_k\varphi_{.j} + \dot{x}^i\varphi_{.j.k}, \tag{4.2}$$

where φ is a function of x and $\dot{x}$ homogeneous of the first degree in $\dot{x}$. The conditions of integrability of (4.2) are

$$\begin{aligned}\zeta^h K^i_{jkl,h} + \zeta^h_{,m}(\dot{x}^m K^i_{jkl,h} - \delta^i_h K^m_{jkl} + \delta^m_j K^i_{hkl} + \delta^m_k K^i_{jhl} + \delta^m_l K^i_{jkh}) \\ = \delta^i_k\varphi_{.j.l} - \delta^i_l\varphi_{.j.k} + \delta^i_j(\varphi_{.k.l} - \varphi_{.l.k}) + \dot{x}^i(\varphi_{.j.k.l} - \varphi_{.j.l.k})\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\zeta^h\Gamma^i_{jkl,h} + \zeta^h_{,m}(\dot{x}^m\Gamma^i_{jkl,h} - \delta^i_h\Gamma^m_{jkl} + \delta^m_j\Gamma^i_{hkl} + \delta^m_k\Gamma^i_{jhl} + \delta^m_l\Gamma^i_{jkh}) \\ = \delta^i_j\varphi_{.k.l} + \delta^i_k\varphi_{.l.j} + \delta^i_l\varphi_{.j.k} + \dot{x}^i\varphi_{.j.k.l}.\end{aligned} \tag{4.4}$$

By contraction we can obtain expressions for $\varphi_{.k.l}$, $\varphi_{.k,l}$ and $\varphi_{.j.k.l}$ in terms of ζ^h, $\zeta^h_{,i}$ and $\varphi_{.i}$. Therefore,

A necessary and sufficient condition that a space must satisfy in order to admit projective collineations is that (4.3) and (4.4) and the equations obtainable from them by differentiation and elimination be compatible for the determination of the $n^2 + 2n$ functions ζ^h, $\zeta^h_{,i}$ and $\varphi_{.i}$.

Evidently the greatest number of linearly independent projective collineations that a space may admit is $n^2 + 2n$; this will be the case when, and only when (4.3) and (4.4) are identities in ζ^h, $\zeta^h_{,i}$ and $\varphi_{.i}$.

The further discussion of collineations, as well as the group properties of collineations and motions, will be presented in a later paper.

[1] Finsler, P., Dissertation, Göttingen, 1918.

[2] Synge, J. L., *Trans. Amer. Math. Soc.*, **27**, 1925, pp. 61–67.

[3] Taylor, J. H., *Trans. Amer. Math. Soc.*, **27**, 1925, pp. 246–264.

[4] Berwald, L., *Math. Zs.*, **25**, 1926, pp. 40–73.

[5] Eisenhart, L. P., *Riemannian Geometry*, **1926**, p. 234.

[6] Knebelman, M. S., these PROCEEDINGS, **13**, 1927, pp. 396–400. Also Eisenhart, L. P., and Knebelman, M. S., *Ibid.*, **13**, 1927, pp. 38–42.

[7] Veblen, O., and Thomas, T. Y., *Trans. Amer. Math. Soc.*, **25**, 1923, pp. 551–608.

FELIX KLEIN AND THE HISTORY OF MODERN MATHEMATICS

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For more than a quarter of a century Felix Klein was a Foreign Associate of this Academy, and during this period no other foreign mathematician exerted a greater influence on the development of mathematics in our country. This influence extended from investigations relating to various advanced branches of this subject to the teaching and history of the elementary parts thereof. His method of work was in unusually close accord with the motto of a younger large American research organization, viz., "Companions in Zealous Research." He imparted his fruitful notions very freely, especially to his students, and it is now impossible in many cases to determine just what was contributed by Klein himself. Even when his "Collected Works" were prepared for the press he discussed various parts thereof in his University lectures and then added thereto illuminating notes arising from these discussions.

His own work illustrates the fundamental principle of the history of mathematics that the great forward movements are due more and more to collective efforts, and that one of the most effective services which an individual can render is to point out resultants of such collective forces and to work in harmony therewith. This is the work of the scientific statesman, and in the mathematical world Klein was preëminently such a statesman. He took a leading part in an effort to collect and coördinate the extensive body of mathematical advances in the form of large encyclopedias, and during the last few years of his life he gave courses of lectures on the development of mathematics during the nineteenth century. While his work along this line was left in an incomplete form it was in a sufficiently advanced state to exhibit his views on the general methods which

should be pursued in such a history. He aimed to popularize mathematics even at the risk of a kind of *pia fraus*, due to making the subject appear easier than it really is. Just as in the case of his treatment of elementary mathematics, he proceeded in his history mainly from the higher point of view instead of toward the higher point of view in the sense of first laying an elaborate basis on which a superstructure leading to this point of view might be erected.

In view of the fact that a treatment of the history of mathematics since the beginning of the nineteenth century presents such great difficulties Klein's method of dealing with this subject will doubtless receive much attention. What is still more important is that he expressed approval of separate efforts along this line by his own example. The mathematical work of the last century has furnished many evidences of the fact that the successful investigator in one field is not necessarily familiar with many of the other fields. Mathematics is not a world empire where one man can exert a dominating influence over work along all the lines, but it is continually splitting up into self-determining republics. There is a growing need for conferences dealing with common affairs of such republics and these common affairs must constitute one of the main elements of a general history of modern mathematics if such a history is to establish closer contact not only between the mathematicians engaged in widely different fields of research but also between the mathematicians and other scientists. The preservation of such contact is largely dependent upon the recognition of serious efforts along this line.

In the introduction to his posthumous work entitled *Vorlesungen über die Entwicklung der Mathematik im* **19**, *Jahrhundent*, 1926, Klein noted that this history did not aim to take the place of a brief mathematical encyclopedia in which the main divisions of the subject are selected and presented in a systematic form. He aimed much more to give selected sketches of the work of eminent individuals and of the purposes and results of definite schools. He explicitly disclaimed completeness of any kind whatsoever, and stated that he did not expect to make minute preparatory studies relating to the subjects treated. He aimed merely to give a tolerably accurate presentation.

These modest claims were doubtless inspired by the fact that a history of the development of mathematics during the nineteenth century from such a standpoint would serve a very useful purpose, since it would enable the reader to see something of the nature and purposes of modern mathematical work without making heavy demands on his attainments. On the other hand, it would naturally inspire others to supply desirable additions and modifications, and thus be conducive to mathematical progress. Such a modification was suggested already by the editors of the work in question in a note to the introduction thereof. It is stated here that

one finds frequently in the writings of Euler remarks which aim to awaken on the part of the reader the desire to advance by his own efforts beyond the work in hand, while Klein claimed that such remarks belonged entirely to writings of the nineteenth century, and referred to the writings of Monge, Jacobi and Faraday as illustrative of this point.

A somewhat similar modification applies to the fundamental statement found on page 84 of the same work to the effect that the arithmetization of mathematics was begun by A. L. Cauchy. In an article which appeared in these PROCEEDINGS[1] about two years ago it was noted that this arithmetization dominated various fundamental mathematical developments since the times of the ancient Greeks. In fact, it appears already in the arithmetical books of Euclid's "Elements." Recently a well-known German mathematical historian supported this view.[2] Such principles of the history of mathematics should be formulated as accurately as possible since they enable the student to associate harmoniously many historical facts, and thus to simplify greatly his mastery thereof. Moreover, the various illustrations of the same general principle tend to deepen insight into the nature of the subject, and such deepening is naturally attended by increasing zest. It, therefore, seemed desirable to refer here to the exact status of the principle in question, especially since it has not yet been widely accepted.

Klein's work relating to the history of mathematics is by no means confined to the lectures on this subject to which we referred above. About twenty years before the publication of these lectures one of his students presented a doctor's dissertation on the history of mathematics at Göttingen. Moreover, Klein frequently referred to fundamental historical questions in his writings. For instance, in an article entitled "Grenzfragen der Mathematik und Philosophie," 1906, he said that in common with all the other scientists the mathematicians had begun to doubt everything that was supposed until then to have been fully established, and that everything was then in fermentation. Such doubt obviously relates only to questions of postulates. Otherwise, the mathematician still has the comfortable feeling that many of the contributions which he is able to make will abide, especially if wide contact can be maintained. In view of the common interest in the origin of useful ideas the history of mathematics seems now to present one of the most effective avenues for maintaining such contact, and as Klein was especially interested in this contact he naturally devoted much thought to this subject during the closing years of his influential life. As scientific life becomes more complex the old avenues of intercommunication should not only be improved but new ones should be sought.

[1] G. A. Miller, *Proc. N. A. S.*, **11**, 546 (1925).

[2] H. Wieleitner, *Jahresbericht D. Math. Vr.*, **36**, 74 (1927).

THE RESIDUAL SET OF A COMPLEX ON A MANIFOLD AND RELATED QUESTIONS

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1. The questions treated in this paper found their starting point in my recent investigations on complexes and manifolds.[1] Among the neighborhoods of a C_k on an M_n there are some whose complementary set is itself an M_n with a boundary in the strict sense of *Tr.* 2, p. 437. The duality theorems of that paper (p. 449) become applicable to them and by relatively simple considerations lead to the two basic duality formulas (3), (4) with their extensions in various directions; (3) is the direct generalization of Poincaré's noted duality relation and of the second duality theorem of *Tr.* 2 (p. 450); (4) includes and extends Alexander's important duality theorem for a C_k on a hypersphere,[2] and its recent generalization by Alexandroff.[3] Alexander's work is the first on the sets $M_n - C_k$ and it is quite probable that his method is applicable to our situation as well. Our attack, however, has been from a different angle and seems decidedly interesting for its own sake. We conclude our discussion with extensions of the coincidence formulas of *Tr.* 1, 2, to residual sets on an M_n.

2. *The Neighborhood of C_k on C_n.*—We take as in *Tr.*1 a C_n whose cells are simplicial and at first C_k a subcomplex of C_n.[4] The cells of C_n abutting on C_k constitute a subcomplex C_n' of C_n, whose cells not abutting on C_k constitute a C_{n-1}. Let PQ be a segment whose end-points P, Q, are, respectively, on C_k and C_{n-1}. Through every point R of C_n' not on C_k or C_{n-1} there passes a unique PQ. To show it we need only consider the case where R is on a certain simplex, or which is the same to assume that C_n is a simplex. Then C_k is a simplex on its boundary, and the property in question goes back to this: Given on an S_n, two independent spaces S_k, S_{n-k-1}, through every point R not on one of them there passes one and only one line intersecting both.

P will be called the *projection* of R on C_k. When R tends to a point on C_k, P tends to the same point, as readily shown by the same consideration as above. Therefore, the points of C_k are to be considered as their own projections on the complex.

3. Given PQ, take for R the point that divides the segment in the ratio $\lambda:1$, $0 < \lambda < 1$. When PQ remains on a definite cell E_h of C_n', R generates a certain E_{h-1}' which is convex and flat and the set of all these cells determines a complex Φ_{n-1}, polyhedral, but with cells that may not be simplicial. In this connection see *Tr.* 1, p. 10.

The set of all segments PR plus their end-points is an n-complex which

we shall denote by $N(C_k)$ or simply N. It is a *closed neighborhood* of C_k, the open neighborhood being $N - \Phi$. The *width* δ of N is the upper bound of PR, and can obviously be made as small as we please. We shall also introduce as of great importance in the sequence the n-complex $K_n = C_n - C_k - N + \Phi$. Regarding the complexes just introduced we have several important theorems.

4. I. *The complexes N, K, Φ remain homeomorphic to themselves when λ varies.*

II. *Every cycle Γ_μ of N is $\sim$ mod. N to a cycle Γ'_μ on C_k and when $\Gamma_\mu \sim 0$ mod. N, then $\Gamma'_\mu \sim 0$ mod. C_k.*

Let C_μ be a complex on N whose cells are oriented and may be singular.[5] As a point R ranges over C_μ its projection P on C_k will define cell for cell a complex C'_μ continuous image of C_μ, the incidence relations being the same between corresponding cells. Hence in the Poincaré congruence notation, from $C_\mu \equiv 0$, follows $C'_\mu \equiv 0$, or the image of a cycle is a cycle. Furthermore, the locus of PR is a $C_{\mu+1}$ whose cells can be so oriented that $C_{\mu+1} \equiv C_\mu - C'_\mu$, hence $C_\mu \sim C'_\mu$ mod. N. Finally, if C_μ, not a cycle, has the boundary $\Gamma_{\mu-1}$, i.e., if $C_\mu \equiv \Gamma_{\mu-1}$, then $C'_\mu \equiv \Gamma'_{\mu-1}$ image of $\Gamma_{\mu-1}$, that is, from $\Gamma_{\mu-1} \sim 0$ mod. N, follows $\Gamma'_{\mu-1} \sim 0$ mod. C_k.

III. *Let K'_n, K''_n correspond to λ', λ'', and let Γ'_μ be a cycle of the first deformed into Γ''_μ when λ varies from λ' to λ''. Then $\Gamma'_\mu \sim \Gamma''_\mu$ mod. the complex common to K'_n and K''_n, and a fortiori mod. $C_n - C_k$.* A similar statement holds for the neighborhood complexes N.

For the difference of the cycles bounds the $C_{\mu-1}$ locus of Γ'_μ.

IV. *The cycles of K_n and of $C_n - C_k$ are in (1,1) correspondence with preservation of their homologies.*

Compare $C_n - C_k$ with K'_n fixed. Every cycle of the latter is also one of the former, and if ~ 0 mod. K'_n, it is likewise ~ 0 mod. $C_n - C_k$. Conversely, Γ_μ of $C_n - C_k$ is on some K''_n, and in the deformation of the latter into K'_n will be reduced to Γ'_μ on K'_n. Furthermore, let there be on $C_n - C_k$ a $C_{\mu+1} \equiv \Gamma_\mu$. Then there exists a K''_n carrying $C_{\mu+1}$. The above deformation will reduce it to a $C'_{\mu+1}$ of K'_n, $\equiv \Gamma'_\mu$; hence $\Gamma'_\mu \sim 0$ mod. K'_n.

V. *The incidence relations between the cells of Φ_{n-1} are the same as between the cells of C_n that carry them.*

VI. *When C_n defines a manifold so do N, K_n, Φ_{n-1}.*

The manifold conditions are as in *Tr.* 2, p. 438. For Φ the theorem follows from V. For K and N we need only verify the conditions in regard to the cells on Φ. The incidence relations being the same for K and N in regard to these cells we need only take up N. Any cell E_j of Φ is carried by an E_{j+1} of C_n. The cells of $N - \Phi$ incident with E_j, other than E_{j+1} itself are of the same dimensionality and have the same incidence relations as those of C_n incident with E_{j+1}. Hence, by condition (*a*) loc. cit., they

have the same incidence relations as the respective cells of $j + 2$ less dimensions of a cellular subdivision of a sphere of S_{n-j-1}. Hence, together with E_{j+1} they constitute a set of cells with the same incidence relations as a subdivision of an E_{n-j-1}, with the cells of Φ incident with E_j corresponding to the boundary cells of the subdivision. Hence, (*b*) loc. cit. is satisfied as to E_j and N.

VII. *The preceding properties can all be carried over to any polyhedral* C_k *on* C_n.

All that is necessary is to replace C_n by a subdivision of which C_k is a subcomplex (*Tr.* 1, p. 10, lemma II).

5. *Connection Indices and Similar Invariants.*—In our previous papers we have only considered cycles on complexes. Actually we may speak of cycles and their homologies in regard to any point set G for which the limit process has a meaning (Fréchet $\mathfrak{L}$ space), and that is more than ample for our purpose. Thus in IV we legitimately considered the cycles of $C_n - C_k$, which is not a complex. See in this connection, Alexander, loc. cit., p. 339.

It will be convenient to denote the μth connection index of G by $R_\mu(G)$. When G is not a complex this index need not necessarily be finite.

Recently Alexandroff[6] and Vietoris[7] have given a definition of the numbers $R_\mu(G)$ and similar invariants, also applicable to sets of a very general type. We leave aside the question of their equivalence to the invariants as defined directly by means of the cycles on the sets.

6. It is advisable to introduce special notations for certain simultaneous topological invariants of a set G and its subset H. They come up naturally in these investigations and their relations to be given below are obtained by considerations such as those of *Tr.* 2, p. 440. These invariants are all maximum numbers of independent μ-cycles, respectively:

(*a*) Of H independent mod. G or $R_\mu\ (H;\ G)$
(*b*) —— dependent ——— $r_\mu\ (H;\ G)$
(*c*) —— independent mod. G of the cycles of $G - H$ or $S_\mu\ (H;\ G)$
(*d*) —— dependent ——— $s_\mu\ (H;\ G)$
(*e*) Of G independent ——— and also of those of H, or $T_\mu(H;\ G-H)$.

T is also the maximum number of independent μ cycles of G of which every linear combination meets both H and $G - H$. We have

$$R_\mu(H;G) + r_\mu\ (H;G) = S_\mu(H;G) + s_\mu\ (H;G) = R_\mu(H);$$
$$T_\mu(H;G-H) = T_\mu(G-H;H).$$

Also $s_\mu\ (H;G) - r_\mu\ (H;G)$ = the same number for $G - H$, since both represent the maximum number of μ cycles of one of the two sets,

homologous mod. G to one on the other, and independent mod. G. Finally we have the fairly obvious formula:

$$R_\mu(H;G) + R_\mu(G - H;G) - [\, s_\mu(H;G) - r_\mu(H;G)] + T_\mu(H;G - H) = R_\mu(G).$$

7. In terms of the preceding invariants the second and first duality theorems of *Tr.* 2., pp. 449, for an M_n with the boundary F_{n-1} are respectively expressed by

$$S_\mu(M_n - F_{n-1};\ M_n) = S_{n-\mu}(M_n - F_{n-1};\ M_n). \tag{1}$$

$$s_\mu(M_n - F_{n-1};\ M_n) = r_{n-\mu-1}(F_{n-1};\ M_n). \tag{2}$$

(1) *represents the true generalization of the Poincaré duality law* and reduces to it for an M_n without boundary.

8. *Duality Relations for* $M_n - C_k$.—Returning to our main topic we now assume that C_n defines an orientable manifold, without boundary, M_n. We continue to take for C_k a polyhedral complex. Then K likewise defines a manifold with the boundary Φ_{n-1}. Applying to it formula (1) together with IV of No. 4 we find immediately

$$S_\mu(M_n - C_k;\ M_n) = S_{n-\mu}(M_n - C_k;\ M_n). \tag{3}$$

This is the duality relation for the cycles of $M_n - C_k$ that do not depend upon those of C_k.

Less immediate is the following formula derived from (2):

$$s_\mu(M_n - C_k;\ M_n) = r_{n-\mu-1}(C_k;\ M_n) + T_{n-\mu}(C_k;\ M_n - C_k), \tag{4}$$

which we shall now prove. Applying (2) to K we have

$$s_\mu(K_n - \Phi_{n-1};\ K_n) = r_{n-\mu-1}(\Phi_{n-1};\ M_n). \tag{5}$$

Let Γ^i_μ $(i = 1,2,\ldots h)$ be independent cycles of $M_n - C_k$, $\sim$ mod. M_n, to cycles Δ^i_μ on C_k. We can so choose K that $K - \Phi$ will carry all the Γ's. Let $C^i_{\mu+1} \equiv \Gamma^i - \Delta^i$. Taking the intersection with Φ,[8] we have a $\mu - 1$ cycle $C^i.\Phi \sim \Gamma^i$ mod. K. Moreover, the cycles $C^i.\Phi$ are independent not only mod. Φ but also mod. K for $\Sigma\lambda_i C^i.\Phi \sim 0$ mod. K leads to $\Sigma\lambda_i\Gamma^i \sim 0$ mod. K or *a fortiori* mod. $M_n - C_k$. Hence to h cycles such as the Γ's correspond as many μ cycles of Φ independent mod. K. Conversely, h cycles of Φ independent mod. K are $\sim$ mod. N, hence mod. M_n to as many cycles on C_k. Therefore,

$$s_\mu(K_n - \Phi_{n-1};\ K_n) = s_\mu(M_n - C_k;\ M_n). \tag{6}$$

So much for the left side of (5). Concerning the other side we have to investigate the $(n - \mu - 1)$ cycles of Φ that bound on K. Such a cycle, $\Gamma_{n-\mu-1}$, may be of one or two types:

(*a*) It is homologous to $\Gamma'_{n-\mu-1}$ on C_k (its projection on C_k), that is, not ~ 0 mod. C_k. Then $\Gamma'_{n-\mu-1} \sim 0$ mod. M_n, but not mod. C_k, so that there exists on M_n but not on C_k a $C_{n-\mu} \equiv \Gamma'_{n-\mu-1}$. Then $C_{n-\mu}.\Phi = \Gamma_{n-\mu-1}$ ($\sim \Gamma'_{n-\mu-1}$ mod. N) bounds the part of $C_{n-\mu}$ on K. Furthermore, $\Gamma_{n-\mu-1}$ is not ~ 0 mod. Φ, else by II of No. 4, $\Gamma'_{n-\mu-1} \sim 0$ mod. C_k. Hence the number of independent Γ 's mod. Φ is the same as that of the number of independent Γ' 's mod. C_k, or $r_{n-\mu-1}(C_k; M_n)$.

(*b*) $\Gamma_{n-\mu-1}$ is $\sim$ mod. N to $\Gamma'_{n-\mu-1} \sim 0$ mod. C_k. We then have $C_{n-\mu}$ on N and $C'_{n-\mu}$ on K such that $C_{n-\mu} \equiv \Gamma_{n-\mu-1}$; $C'_{n-\mu} \equiv \Gamma'_{n-\mu-1}$. Hence, $\Gamma_{n-\mu} = C'_{n-\mu} - C_{n-\mu}$ is a cycle that intersects Φ into $\Gamma_{n-\mu-1}$.

Now we can select a set of independent $(n - \mu)$ cycles of M_n, in maximum number $R_{n-\mu}(M_n)$, consisting of $T_{n-\mu}(C_k; M_n - C_k)$ cycles $\Gamma^i_{n-\mu}$ $(i = 1,2,\ldots T)$ independent of the cycles on C_k or on $M_n - C_k$, and of $p = R_{n-\mu}(M_n) - T$ cycles $\Delta^i_{n-\mu}$ $(i = 1,2,\ldots p)$ some of which are on C_k, some on $M_n - C_k$. Then

$$\lambda\Gamma_{n-\mu} \sim \Sigma\lambda_i\Gamma^i_{n-\mu} + \Sigma\mu_i\Delta^i_{n-\mu}, \text{ mod. } M_n; \lambda \neq 0. \qquad (7)$$

Since the Δ's not on C_k do not have any point on it, the distance from them to C_k exceeds a certain $\delta > 0$. Construct N such that all its points are no farther than δ from C_k. Then Φ will not meet any Δ whatever. Hence, from (7) taking the intersections with Φ we find $\lambda\Gamma_{n-\mu}.\Phi = \lambda\Gamma_{n-\mu-1} \sim \Sigma\lambda_i\Gamma^i_{n-\mu}.\Phi$, mod. Φ. Hence $\Gamma_{n-\mu-1}$ depends mod. Φ upon the T cycles under the sum. These cycles on the other hand are independent mod. Φ. For let $\Gamma'_{n-\mu}$ be a linear combination of the Γ^i such that $\Gamma'.\Phi \sim 0$ mod. Φ. We then have on Φ a $C_{n-\mu} \equiv \Gamma'.\Phi$. Now Γ' can be assumed polyhedral and without $n - \mu$ cells on Φ. It can then be decomposed into $C'_{n-\mu} + C''_{n-\mu}$, the first on N, the second on K, with $C' \equiv \Gamma'.\Phi$, $C'' \equiv -\Gamma'.\Phi$. Also $\Gamma' = (C' - C) + (C'' + C)$. Each parenthesis represents a cycle, the first on N; hence, $\sim$ mod. N to a cycle on C_k, the second on $M_n - C_k$. But Γ', linear combination of the Γ^i, cannot be decomposed into such a sum, a contradiction which proves that the $\Gamma^i.\Phi$ are independent mod. Φ. The number of $n - \mu - 1$ cycles of the type now considered is then, like that of the Γ^i, equal to T.

Adding up the numbers of cycles of the two types (*a*), (*b*), we find that the right sides of (4) and (5) are equal. Then from (6) follows (4) whose proof is thus complete.

Remark. The proof fails when Φ, K, hence $M_n - C_k$, are vacuous sets. Therefore, it is essential that C_k be a true subset of M_n.

9. *Extension to an Arbitrary C_k on M_n.*—We now assume that C_k is any *non-singular* complex on M_n, not homeomorphic to M_n, so that $M_n - C_k$ is a non-vacuous set, i.e., there are points of M_n not on C_k. This is scarcely a restriction since when C_k coincides with M_n the questions here considered lose all interest. The basis of the extension is the theorem and corollary

to be proved below, natural generalizations of proposition II of No. 4:

Theorem: *Let N_δ be the set of all points of M_n whose distance from C_k is $< \delta$. If C_μ is any complex (singular or not) on N_δ whose boundary is on C_k, then for δ sufficiently small there is a C'_μ on C_k, such that $C_\mu \sim C'_\mu$ mod. N_δ.*

The proof is quite simple. For $k = 0$ the theorem reduces to II, No. 4 itself. Granting it for $k-1$ we shall show that it holds for k. Let C_{k-1} be the subcomplex of C_k where it fails to behave like an M_k. Subdivide C_μ into cells of diameter $< \epsilon$ sufficiently small and apply a certain construction due to Alexander,[9] his B vertices being now on C_k and the cells that they determine not necessarily straight but merely cells on C_k. The construction merely requires that δ, ϵ, be sufficiently small. The effect will be to have on N_δ a $C_{\mu+1} \equiv C^1_\mu - C^2_\mu + C^3_\mu$, where C^1 is on C_k, C^2 is very near C_{k-1} and C^3 includes all cells of C_μ, whose points are at a distance exceeding a certain α sufficiently small from C_{k-1}. Then $C_\mu - (C^1 - C^2 + C^3)$ which is $\sim C_\mu$ mod. N_δ, can be written as $C^2_\mu + C^4_\mu$, where C^4 can be brought as near as we please to C_{k-1} by taking δ, ϵ, α, small enough. The common boundary $\Gamma_{\mu-1}$ of C^2 and $-C^4$ is a cycle on C_k very near C_{k-1}, hence by II, No. 4, we can write a congruence $C^5_\mu \equiv \Gamma_{\mu-1} - \Gamma'_{\mu-1}$ where C^5 is on C_k as near as desired to C_{k-1} and Γ' is on C_{k-1}. Now $C^2 + C^4 = (C^2 - C^5) + (C^5 + C^4)$. The first parenthesis represents a complex on C_k, the second a complex whose boundary is on C_{k-1} and as near to the latter as we please. Since the theorem holds by assumption for $k-1$, $C^5 + C^4$ will be $\sim$ mod. a neighborhood of C_{k-1}, which we may manage to choose on N_δ, to a C^6_μ on C_{k-1} itself, hence on C_k. It follows finally $C_\mu \sim C^2 - C^5 + C^6$, mod. N_δ where C^2, C^5, C^6 are on C_k. The theorem is, therefore, proved.

Corollary: Under the same circumstances as in the theorem every cycle on N_δ is $\sim$ mod. N_δ to a cycle on C_k. This is II, No. 4 for our type of complex.

In order to apply properly the results of *Tr.* 2, in particular to derive (3), (4), for our C_k, we need in place of N_δ, a neighborhood which together with its boundary constitutes a manifold. Its construction is very simple. Subdivide M_n into cells whose diameter is $< {}^1/_2\,\delta$; take all n-cells carrying a point of C_k plus their boundaries, and construct a neighborhood of the C_n to which they give rise in the same manner as in No. 3, taking care that Φ_{n-1} be within a distance ${}^1/_2\,\delta$ of the C_n. What we naturally denote now as $N(C_k)$ is a manifold which consists of the C_n plus its neighborhood $+ \Phi$, and $K_n = M_n - N + \Phi$. Thanks to the Theorem and Corollary of No. 9 the proofs of (3), (4) go through without a change. Therefore, *these formulas stand proved for the most general (non-singular) C_k on M_n, actual subset of M_n.*

11. *Further Extensions.*—On examining the derivation of our formulas

it will be found that it is valid when M_n is replaced by a C_n, and C_k by a subset G of C_n such that: (*a*) G is a true subset of C_n. (*b*) There is a neighborhood of G whose points are all within a distance δ arbitrarily small of G, such that its complementary set K_n is an M_n. (*c*) The theorem and corollary of No. 10 are valid. We shall consider several important cases.

I. Let C_n satisfy the manifold conditions for all cells except those of a subcomplex C_k, $k < n$. Then (3), (4) will be applicable with M_n replaced by C_n, for (*a*), (*b*) hold, and (*c*) reduces to II, No. 4. Let, in particular, C_k be the boundary F_{n-1}. Then (3) turns directly into (1) except for C_n in place of M_n. However, in place of (2) we have

$$s_\mu(C_n - F_{n-1};\ C_n) = r_{n-\mu-1}\ (F_{n-1};\ C_n) + T_{n-\mu}\ (F_{n-1};\ C_n - F_{n-1}). \quad (8)$$

Thus, when a C_n behaves like an M_n everywhere except at the boundary, a new term appears at the right in (2). In this connection we recall that Veblen has practically defined an M_n with boundary, as a subcomplex of an M_n without boundary.[10] For such an M_n, (1) holds but (2) must be replaced by the more involved (8).

An interesting example of the type of complex here considered is an algebraic surface or variety with its locus of singular points.

II. G is the homeomorph of a so-called *analytical* manifold (defined by a set of analytical expansions of the coördinates) or obtained by piecing a finite number of such manifolds. The boundaries of the pieces are assumed of the same type as G.[11]

III. Our formulas are valid when G is an *arbitrary closed* (true) subset of M_n, provided that cycles, complexes, etc., by means of which the various invariants are defined, are interpreted thus: A sequence of μ-cycles $\{\Gamma_\mu^p\}$ defines a μ-cycle Γ_μ on G if for every δ sufficiently small there is a p_0 such that all cycles Γ^p with $p > p_0$ are on N_δ and homologous to each other mod. N_δ. By definition $\Gamma \sim 0$ mod. G, if $\Gamma^p \sim 0$ mod. N_δ for every $p > p_0$. From this to the comparison of cycles, and the new definitions of the invariants of No. 6, is but a step and it is clear that all the elements for the extension of our formulas are at hand.

The cycles and connection indices so defined are the same as those recently introduced by Alexandroff[6] and Vietoris.[7] Indeed for δ sufficiently small Γ^p, $p > p_0$, can be reduced by Alexander's construction to a complex whose vertices are *on* G. Furthermore, if on N_δ there is a $C_{\mu+1} \equiv \Gamma_p$, $C_{\mu+1}$ and the cycle can simultaneously be reduced to a complex and cycle ($\sim \Gamma^p$ mod. N_δ) whose vertices are on G. Then the identity with the Vietoris process becomes apparent.

IV. The extension to a set G on an M_n with the boundary F_{n-1} offers no difficulty, provided G be replaced by $G + F_{n-1}$ and the formulas interpreted accordingly.

12. An extension of a different nature is obtained by considering with Alexander,[12] Poincaré congruences and homologies whose coefficients are reduced modulo a prime number p. Then C_μ is a cycle if its boundary is of the form $pC_{\mu-1}$; $C_\mu \sim 0$ if there is a C'_μ such that $C_\mu + pC'_\mu \sim 0$ in the usual sense. The various indices are then defined as in No. 6. Our formulas are then preserved.[13] This is likewise the case for congruences and homologies mod. 2 introduced earlier by Veblen and Alexander (see Veblen, loc. cit., Ch. III). They give rise to their so-called *connectivities*, which are $1 + R_\mu(G)$ in our notation (mod. 2, of course). In that case and with $M_n = H_n$, a sphere of S_n (4) turns into Alexander's main result,[2] the first along the lines investigated in this Note. It also includes the recent generalization given by Alexandroff.[3] Indeed, let G be any subset of H_n; $R(G;H) = S_\mu(\text{—}) = 0$, $T_\mu = 0$ unless $\mu = n$, when its value is 1; $r_\mu(\text{—}) = s_\mu(\text{—}) = R_\mu(G)$. Hence, if G is any closed set to which the formulas apply, (3) disappears and (4) becomes

$$R_\mu(H - G) = R_{n-\mu-1}(G) + \begin{cases} 0 \text{ for } \mu < n \\ 1 \text{ for } \mu = n \end{cases}, \qquad (9)$$

where the indices can also be taken mod. p. The reasoning by which Alexander derives the theorem of Jordan-Brouwer from his formula is also applicable to (9).

13. *Continuous Transformations.*—Let C_n be a complex with a closed subset F that includes in all cases the boundary of C_n, and such that for all elements of C_n without points on F the manifold condition is verified. Thus, C_n may be a manifold with boundary in the sense of Veblen. Let the indices $R_\mu(F)$ in the generalized sense of Alexandroff-Vietoris be finite. Then we can find a neighborhood N of F with the corresponding $K_n = C_n - N + \Phi$ as in No. 10, to which the formulas for coincidence and fixed points of *Tr.* 2 can be applied almost directly. V_n and the cycles Γ, Δ, D, G of *Tr.* 2, pp. 439–441 are defined as in *Tr.* 2 with these reservations: (*a*) The G's are independent cycles of $C_n - F$ (not necessarily of F) homologous mod. C_n to cycles on F. (*b*) A skew-symmetric cycle is a Δ if, and only if its intersection with F (base on F) is ≈ 0 mod. N (not the more stringent mod. F). Then there will be a corresponding skew-symmetric cycle associated with V_n of K_n. It will be found that the whole derivation of the coincidence and fixed point formulas given in *Tr.* 2 for an M_n with boundary is applicable to $C_n - F$.

As an interesting special case let C_n be a manifold in the sense of Veblen with $F = F_{n-1}$ as its boundary, and let it be susceptible of a continuous deformation into a set on $C_n - F$ (deformation into the interior of itself). Then if a cycle of F is ~ 0 mod. C_n it is deformed into a homologous cycle on $C_n - F$, which is ~ 0 mod. $C_n - F$, with a similar remark for the base of a skew-symmetric cycle on F. Therefore, the interpretation

of the G's and Δ's can be taken strictly as in *Tr.* 2. From this follows that (37.1) of *Tr.* 2 represents again the number of signed fixed points of a *continuous* deformation and *it is then in this case also equal to the Euler-characteristic of the complex.*

[1] Lefschetz, *Trans. Amer. Soc.*, **28,** 1926 (1–49); **29,** 1927 (429–462); Respectively referred to as *Tr.* 1, *Tr.* 2. Further references are given at the beginning of *Tr.* 1. The notations and terminology of this Note are as given at the beginning of *Tr.* 1. I take advantage of this occasion to point out the following errata for *Tr.* 2: Page 442, lines 5, 6, 8 replace μ by $\mu - 1$. Page 449, line 6, replace $n - \mu$ by $n - 1 - \mu$.

[2] Alexander, *Trans. Amer. Soc.*, **33,** 1922 (333–349).

[3] Alexandroff, *Comptes Rendus*, **184,** 1927 (425–427).

[4] As a matter of fact C_n and C_k may be any agglomeration of simplicial cells, no two of which have any common point, that constitutes a closed set. This is what Alexander (loc. cit.) calls a *chain*. The indices are then the maximum dimensions of cells of the set; a similar statement applies to C'_n, and repeatedly in the sequence.

[5] See *Tr.* 2, p. 5; also Veblen, *The Cambridge Colloquium*, New York, **1922,** p. 74.

[6] Alexandroff, *Comptes Rendus*, **184,** 1927 (317–319).

[7] Vietoris, *Math. Ann.*, **97,** 1927 (454–472).

[8] The rigorous justification for our seemingly reckless use of intersections of various cycles and complexes throughout this paper has been given at length in *Tr.* 1, 2.

[9] Alexander, *Trans. Amer. Math. Soc.*, **26,** 1915 (148–154).

[10] Loc. cit., p. 88.

[11] However, I do not think that it would be difficult to prove that these analytical manifolds are complexes.

[12] Alexander, *Trans. Amer. Math. Soc.*, **28,** 1926 (301–329).

[13] In the chain of arguments leading to formulas (3), (4) (*Tr.* 2 down to the duality theorems and present Note), the only step whose extension is not automatic is the proof of Veblen's theorem (*Trans. Amer. Math. Soc.*, **25,** 1923 (540–550); also *Tr.* 2, p. 433): There can be found for an orientable M_n without boundary two associated sets of cycles Γ^i_μ, $\Gamma^j_{n-\mu}$ $(i, j = 1, 2, \ldots, R_\mu)$, whose Kronecker-index matrix $||(\Gamma^i_\mu . \Gamma^j_{n-\mu})||$ is $\pm$ the identity matrix of order R_μ. Now, in Veblen's notation, an independent set mod. p is obtained by adding to his circuits Γ_μ: (*a*) The complexes Δ^i_μ whose coefficient of torsion t is a multiple of p; (*b*) the complexes Ψ^i_μ whose boundary is a $\Delta_{\mu-1}$ of the preceding type. But from Veblen's formulas, loc. cit., p. 546, coupled with his remarks, p. 547, follows that his theorem is still valid for the associated sets mod. p: Γ^i_μ, Δ^i_μ, Ψ^i_μ; $\Gamma^i_{n-\mu}$, $\Psi^i_{n-\mu}$, $\Delta^i_{n-\mu}$, where now R_μ is the connection index mod. p. The Kronecker indices may, of course, be reduced mod. p. This justifies the assertion of the text. Incidentally Veblen's theorem implies $R_\mu \leqq R_{n-\mu}$, $R_\mu \geqq R_{n-\mu}$, hence $R_\mu = R_{n-\mu}$ which is Poincaré's duality relation for indices mod. p of an M_n without boundary.

The above holds for $p = 2$ as well. Furthermore, from Veblen's remarks at the end of his paper, it appears that the results of *Tr.* 1, 2, and of the present Notes, are applicable to non-orientable manifolds, when congruences, homologies and Kronecker-indices are reduced mod. 2.

A DISPROOF OF THE RADIATION THEORY OF CHEMICAL ACTIVATION

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Arrhenius explained the usual change of reaction rate with temperature by assuming that only those molecules react that have acquired a certain minimum energy. This view is now universally accepted. It is also agreed that the abnormally high energy of these activated molecules must be acquired either by collision with other molecules or by the absorption of radiation. There has, however, been some conflict of opinion as to which of these methods of activation is predominant and determines the rate of a reaction.

In a purely photochemical reaction the absorption of radiant energy is plainly responsible for the activation. This suggested the possibility that thermal reactions are also due to activation by the thermal radiation which is present at every temperature.[1] The argument was very forcibly presented by Perrin who showed that if the specific rate of a unimolecular gas reaction remains constant, with indefinite diminution in pressure, activation must be by radiation since the number of opportunities for activation by collision also diminishes without limit. In fact, the decomposition of nitrogen pentoxide, the first gas reaction shown to be unquestionably unimolecular,[2] was found to have a specific reaction rate constant over a wide range of pressure, and apparently increasing at very low pressures.[3]

On the other hand, it was questioned[4] whether even under favorable circumstances the thermal radiation is present in sufficient amount to be able to account for the observed high rate of decomposition of nitrogen pentoxide. However, it was shown by Lewis and Smith[5] that, considering the large thermal energy already possessed by the molecules of this gas, owing to their high heat capacity, activation by radiation might account for the observed rate of reaction, provided that nitrogen pentoxide in a large region of the infra-red has a coefficient of absorption nearly as high as the maximum that is to be regarded as theoretically possible.

However, a study of the absorption of radiation by nitrogen pentoxide made by Daniels,[6] between the visible region and about 8μ, showed absorption coefficients in no part of this range sufficiently great to support the radiation hypothesis. Nevertheless, it occurred to us, since the greater part of the energy of thermal radiation at room temperature lies still farther in the infra-red, and also since Daniels' results showed a greater absorption coefficient in the long than in the short wave-lengths, that the

discovery of a very high absorption coefficient at longer wave-lengths might still permit the maintenance of the radiation hypothesis. We, therefore, attempted to make such an investigation as was possible of the absorption over the whole important spectral range.

At long wave-lengths such a study is extremely difficult. But by dispensing entirely with windows, using a black body at 100°C. and 200°C. as the source of radiation, using a thermopile, as sensitive as it can be made without enclosing it in a vacuum, to receive the radiation, and by placing between the radiation and the thermopile a vessel through which pure dry air, or air mixed with nitrogen pentoxide, could be passed, we were able to obtain rough measurements of the coefficient of total absorption. This proved to be of the same order of magnitude, and certainly not much larger than the absorption coefficient at the longest wave-lengths investigated by Daniels.

As far, therefore, as these rough measurements went they seemed to disprove the hypothesis that the decomposition of nitrogen pentoxide is activated by radiation. It seemed, however, desirable to verify so important a conclusion by experiments of a more trustworthy character. Daniels[7] had already studied the decomposition of nitrogen pentoxide when subjected to external radiation, and found little or no effect. But all experiments of this sort are open to the criticism that the effective portion of the radiation may be largely absorbed before it has penetrated far into the gas, and also that deactivation of partially activated molecules by collisions of the second kind take place so rapidly at normal pressures that the increase in specific rate cannot be distinguished from that due to the general temperature rise of the gas. If, instead of subjecting to external radiation a mass of the gas at atmospheric pressure, we could allow the gas to flow through a radiating tube without coming in contact with the walls, and at such low pressures that there is no screening of one portion of the gas by another and also no collisions, it seemed to us that a conclusive experiment might be made.

We, therefore, decided to study the reaction of a molecular stream passing through an exhausted enclosure which is permeated by radiation corresponding to a temperature that would be sufficient under ordinary circumstances to produce chemical reaction. Since, in the molecular stream, the gas is at so low a concentration that the mean free path is considerably longer than the length of the stream, the number of collisions is negligible. This method seems, therefore, to offer a direct means of investigating the reaction due to radiation alone.

Since we planned to collect on a receiving surface the material passing straight through the molecular stream, and to ascertain from analysis of this portion the fraction reacting, it seemed inexpedient to employ nitrogen pentoxide or any other substance which dissociates into two or

more molecules; for these, flying apart at the moment of dissociation, might not arrive at the collecting surface. We, therefore, decided to use pinene, the racemization of which had been shown by Smith[8] to be one of the most satisfactory unimolecular reactions so far investigated.

The radiation field was produced by a heated platinum wire wound longitudinally on a quartz cylinder, inside and out. The radiation thus produced did not correspond exactly to isothermal black body radiation, since the temperature measured by a thermocouple inside the cylinder was at 1000°K. when the temperature of the wire was 1700°K. But we have been able to estimate that in no significant part of the spectrum was the density of energy within the cylinder less than that which would be present in a Hohlraum at 800°K. Indeed, over the greater part of the spectrum, the energy density at each frequency was greater than that in a Hohlraum at 1000°K. Under the conditions of the experiment the molecules were subjected to radiation for 10^{-4} seconds, and during this period of time, at a temperature of 800°K., the racemization, according to the temperature coefficient of the reaction determined by Smith, would amount to ten per cent. At 1000°K. only the 10^{-6} part would remain unracemized. In our experiment *no racemization could be detected.*

We seem, therefore, to have proved conclusively that, at least in this one reaction, radiation cannot alone be responsible for the process of activation. This experiment, together with the recent observations of Hinshelwood and Thompson,[9] Hinshelwood,[10] Ramsperger[11] and Rice and Ramsperger,[12] which show that typical unimolecular reactions do suffer a diminution in specific reaction rate with decreasing pressure and thus render invalid the powerful argument of Perrin, appears to remove all support from the radiation hypothesis.

A full description of the experiments here mentioned will shortly be submitted by one of the authors to the *Journal of the American Chemical Society.*

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RADIATION AND COLLISION IN GASEOUS CHEMICAL REACTIONS

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The mechanism of the activation process in homogeneous gaseous systems may be broadly divided into two possibilities: (1) activation by radiation, (2) activation by collision. The difficulty has been to devise a mechanism which would adequately provide the energy necessary to maintain the requisite number of molecules in an active state, and which would account for the observed rate of reaction. No proposed explanation has been without some objection.

The original concept of the radiation hypothesis[1] as an explanation of the molecular activation has been subjected to some modification and much criticism[2] chiefly due to a disagreement between the radiation density of the activating frequency calculated from Planck's equation and the critical increment. There is also a disparity between the activating frequency calculated from the temperature coefficient and the absorption bands in most cases. The proposed remedies postulate activation by a continuous range of frequencies, light quanta of large size and other modifications.

Regardless of the particular mechanism, if the activation process is accomplished through the absorption of single or multiple frequencies, then a marked increase in the radiation density of these frequencies should cause a change in reaction velocity. The decomposition of ozone and nitrous oxide have been studied from this point of view and are reported here. Since the inception of this work, it has been found elsewhere[3] that the oxidation of alcohol vapor and the decomposition of nitrogen pentoxide and hydrochloric acid were not affected by infra-red radiation. The present work confirms these results for two additional reactions.

Nitrous oxide and ozone were subjected to infra-red radiation at different temperatures and with different frequencies. These are very divergent types of gas reactions. The decomposition of nitrous oxide was carried out in a quartz vessel at high temperatures, and in a vessel containing a fluorite window at lower temperatures. The radiation density in the infra-red was increased by means of an arc lamp. With the reaction vessel at 883°K., the upper limit of the effective increase is approximately 2.5mu. At this wave-length the input from the external source, after due correction for absorption and unequal emission, is several times that available from the immediate thermal environment. No change in reaction velocity was found. In the case of ozone, the upper limit can be

extended to 9.6mu, the point of maximum infra-red absorption. At this wave-length the corrected external energy input is four times that derived from the immediate surroundings. At shorter wave-lengths this is enormously increased. A certain amount of isolation was possible by means of quartz and biotite screens, thus illuminating the ozone at different periods with different series of possible activating frequencies. Again no observable effect was found. It would seem apparent, therefore, that if there is activation by radiation it is more complex than simple absorption of one or more activating frequencies.

The remaining postulate, activation by collision, has been the subject of considerable speculation.[4] Simple activation by collision has been

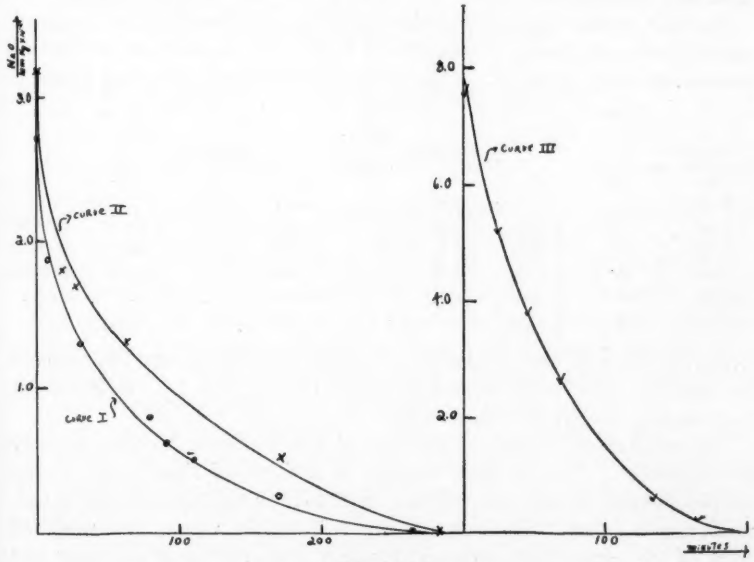

FIGURE 1

found adequate to account for the decomposition of nitrous oxide, hydrogen iodide, chlorine monoxide, acetaldehyde, and the formation of hydrogen iodide,[5] but when applied to unimolecular reactions is wholly inadequate. Various chain,[6] quantum chain[7] and ternary collision[8] theories have been proposed to account for the observed rates of reaction. In addition to the more general theories of unimolecular reaction, several specific suggestions have been advanced. The independence of reaction rate and pressure has been accounted for by means of a time lag between activation and decomposition.[9] However, activation by collision alone will not always maintain an undiminished concentration of active mole-

cules, as required. To surmount this difficulty the assumption has been made that there might be enough collisions if the total energy could be made up by any distribution among many degrees of freedom.[10]

In this complex state of affairs it seemed of interest to examine some reactions at pressures so low that some differentiation between the proposed mechanisms would be possible, or at least some experimental data in a simplified form would be available. It is evident that, at sufficiently low pressures, a profound modification of the velocity constant should take place whether the mechanism be that of Christiansen and Kramers, Lindemann, Perrin or Rodebush. The relation between collision frequency and observed reaction rate at this point would also be of considerable significance.[11]

For this general purpose the decomposition of nitrous oxide, a homogenous bimolecular reaction was chosen first. The results lead to different conclusions than expected. They are presented in the preceding figure 1.

CURVE I 625°C.			CURVE II 550°C.			CURVE III 605°C.		
PRESSURE $N_2O \times 10^{-2}$ MM. HG	TIME MIN.	K	PRESSURE $N_2O \times 10^{-2}$ MM. HG	TIME MIN.	K	PRESSURE $N_2O \times 10^{-2}$ MM. HG	TIME MIN.	K
2.68	...	....	3.17	...	....	7.53	...	....
1.85	6½	0.023	1.81	16.8	0.033	5.20	23.4	0.0161
1.36	30	0.021	1.71	28.0	0.022	2.60	66.6	0.0159
0.80	78	0.0155	1.31	64.0	0.0137	0.53	132.0	0.0195
0.60	90	0.0166	0.51	72.0	0.011	...	...	....
0.26	170	0.0138						

Curves I and II were obtained by reading the total pressure increase in the system. Curve III was obtained by freezing the N_2O in liquid air and reading the N_2 and O_2 formed.

The conclusion is that the reaction at low pressures becomes entirely heterogeneous. The reaction velocity is a *thousand times* that of the simple gaseous bimolecular reaction, and follows the equation for a unimolecular reaction. The reaction was followed by means of a modified form of the Pirani Hale manometer which permitted great accuracy and reproducibility. The heterogeneous reaction must ultimately become zero order or it would otherwise be apparent at higher pressures. Details of the method employed will be given in a subsequent and more extensive publication.

The technique and the reaction system developed for the foregoing reaction was then applied to the study of a unimolecular reaction. The decomposition of nitrogen pentoxide has been found to remain unimolecular over enormous variations in pressure.[12] But what is of most concern to the immediate problem is that it seems to remain unaffected at extremely low pressures. Hunt and Daniels[13] found that the decomposition rate was unchanged even when the partial pressure of nitrogen pentoxide

was reduced to $^1/_{100}$ mm. Hg. However, from the point of view of low pressure effect, this method is without the kinetic significance of a similar study at an absolute pressure of $^1/_{100}$ mm.

Hirst and Rideal[14] have followed the reaction by means of a Pirani Hale manometer at low absolute pressures. Their results have led them to the interesting conclusion that, below a total pressure of 0.25 mm. Hg, the reaction velocity increases, approaching, with decreased pressure, a value of about five times that obtained at high pressures, that is, the equation

$$-\frac{dC}{dT} = 4.98 \times 10^{13}\, e^{-\frac{24{,}700}{RT}}.C$$

becomes

$$-\frac{dC}{dT} = 2.59 \times 10^{14}\, e^{-\frac{24{,}700}{RT}}.C,$$

which corresponds to the Rideal-Dushman equation[15]

$$-\frac{dC}{dt} = \nu e^{-Nh\nu/RT}\, C.$$

This phenomenon is explained by Rideal on the assumption that $^1/_5$ of the molecules undergo decomposition irrespective of the pressure, but that $^4/_5$ undergo decomposition only if they do not after activation suffer collisions with other molecules for a period of 10^{-6} seconds. This presupposes four NO linkages and one shared NO linkage in the pentoxide molecule. For decomposition to occur it is necessary for the energy to pass from one oscillator to another in the same molecule, the time required for this process being of the order of 10^{-6} seconds.

In view of the fundamental nature and importance of these conclusions, it was felt, therefore, that the unimolecular reaction to be studied should be the decomposition of nitrogen pentoxide and at still lower pressures, if possible. The results of Hirst and Rideal have not been duplicated. The velocity constants obtained in the present study are in good agreement with those obtained by Hunt and Daniels. The reaction was studied over initial pressures of N_2O_5 ranging from 18×10^{-2} to 1.8×10^{-3} mm. Hg.

The results are as follows:

TIME (MINUTES)	$N_2O_5 \times 10^{-2}$ MM. HG DECOMPOSED EXPERIMENT I 30.1°C.	K UNIMOLECULAR VELOCITY CONSTANT
32	2.54	0.00447
110	7.0	0.0042
170	9.3	0.0042
275	12.8	0.00422
442	15.8	0.00427
∞	18.6	0.00427

TIME (MINUTES)	$N_2O_5 \times 10^{-2}$ MM. Hg DECOMPOSED	K UNIMOLECULAR VELOCITY CONSTANT
	EXPERIMENT II 30.1°C.	
18	1.00	0.005
44	2.32	0.005
88	4.20	0.0051
139	5.6	0.0048
259	8.0	0.0046
499	10.3	0.0045
∞	11.5	
	EXPERIMENT III 30.1°C.	
20	1.2	0.0056
17	2.1	0.0056
67	3.3	0.0052
130	5.4	0.00506
277	8.2	0.00476
412	9.6	0.0048
877	11.0	0.0046
∞	11.2	
	EXPERIMENT IV 30.1°C.	
11	0.18	0.00562
58	0.82	0.0054
94	1.12	0.00496
196	1.84	0.00486
311	2.30	0.00468
471	2.64	0.0045
931	2.95	0.0044
∞	2.955	
	EXPERIMENT V 40.0°C.	
18	2.5	0.0168
60	6.0	0.0166
105	7.8	0.0165
139	8.5	0.0162
259	9.35	0.0160
∞	9.5	
	EXPERIMENT VI 30.1°C.	
147	0.16	0.0100
∞	0.18	

Using 25,000 as the approximate critical increment, this gives at 30.1°C. the following comparison:

$$K = 0.00424 \text{ calculated.} \quad K = 0.0044 \text{ found.}$$

The final values in a series are taken as the nearest to the correct values since the presence of any foreign matter would tend to make the initial decomposition greater. However, it seems not improbable, in view of a similar phenomenon of the same magnitude in the case of the N_2O decomposition, that this variation may be inherent in the low-pressure system.

In any event it is not to be confused with the change found by Hirst and Rideal as it is not even of a comparable magnitude. Furthermore, a six- to one-hundred fold variation in the initial concentration of the pentoxide produces no appreciable change in the average velocity constant. Last of all the critical increment calculated from the measured velocity constants, 0.0044 at 30.1°C. and 0.0162 at 40.0°C. yields the following result:

CRITICAL INCREMENT CALCULATED	CRITICAL INCREMENT DANIELS AND JOHNSON
24,775	24,700

The agreement is better than the experimental error justifies. In Experiment VI the constant is double the previous values, but, as the error in measuring the oxygen pressure to values of 9×10^{-4}, is not accurate to 20 per cent this is without great significance. A twenty-five per cent error in the pressure measurement at this pressure would double the velocity constant. It nevertheless serves to indicate that there is no great departure in the velocity constant when the initial pressure is reduced to $^1/_{25}$ that of Experiment IV.

Summary.—The mechanism of the activation process in gaseous systems has been investigated from the point of view of (1) activation by radiation (2) activation by collision. An increase in the radiation density of possible activating frequencies has resulted in no increased reaction velocity. The study of the bimolecular decomposition of nitrous oxide at low pressures has led to the conclusion that the reaction is entirely heterogeneous at these pressures. A study of the unimolecular decomposition of nitrogen pentoxide between pressures of $^2/_{10}$ mm. Hg and 2×10^{-3} mm. Hg shows no alteration in the rate of reaction such as was found by Hirst and Rideal but follows exactly the rate determined by Daniels and Johnson at high pressures. No diminution of the reaction velocity as might be exexpected from Lindemann's theory was observed.

The author takes this opportunity to express his appreciation of the assistance given him by Professor H. S. Taylor.

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NOTE ON A CORRECTION FORMULA FOR ARTIFICIALLY DEFORMED CRANIA

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Artificial deformation of the skull is a widespread phenomenon which exists among the primitive people of every continent. Its practice, however, was most extensive among the Indians of America, where all types of deformation are found. Usually artificial deformation is divided into the following classes, based mainly on the method employed.

1. *Occipital.* The pressure is exerted on the posterior surface of the skull to produce a flattened occiput. Cradling may bring about the same moulding. The principally modified diameters of the vault are the cranial length, which is shortened, and the cranial width, which is increased.

2. *Fronto-Occipital.* In addition to the occipital pressure a frontal one is applied. This type is always consciously produced and results in a notably shortened cranial length with an increased width and height.

3. *Annular or Aymara.* This type resembles a cone directed posteriorly. Usually bandages are employed which are placed across the frontal region and encircle the head, passing across the temporals and the occipital bone.

Such distortions of the cranial vault make it impracticable to compare the measurements of deformed with undeformed crania. Consequently for great areas in America, where deformation was almost universal, our progress in determining racial affinities has been impeded. In order

to make available for comparative study such deformed crania the following statistical method has been evolved.

On the basis of the means of 76 groups of undeformed crania, representing all parts of the world, a correlation of +0.65 was determined for basion-nasion and maximum cranial length. In addition, eleven random intra-racial correlations between the same diameters gave coefficients which ranged from +0.42 to +0.63. It was established by a comparison of deformed and undeformed crania from the same sites that although cranial length, breadth and height are markedly modified by deformation, nevertheless the basion-nasion diameter remains unchanged. Therefore, on the basis of the unchanged basion-nasion diameter, it is possible to correct the mean cranial length of the deformed group. The correction used is determined from the regression:

$$r\frac{\sigma x}{\sigma y}$$

$$+0.65\frac{4.41\,x}{1.93\,y}$$

$$x = 2.49\,y.$$

By a similar method the maximum cranial width may be corrected for deformation. In this case the correlation used is between cranial width, which is the diameter to be corrected, and an index obtained by dividing the cranial length into the cranial module. A coefficient of correlation of +0.71 was obtained by using the means of 90 undeformed groups.

$$r\frac{\sigma x}{\sigma y}$$

$$+0.71\frac{5.06\,x}{3.61\,y}$$

$$x = 0.99\,y.$$

Finally, cranial height (basion-bregma) may be corrected by subtracting the sum of the corrected cranial length and width from the sum of the uncorrected cranial length, width and height. This is based on the belief that the cranial module remains unchanged under deformation, but at the present it cannot be completely established because of the lack of homogeneous deformed and undeformed series from the same site. However, another method, which is tentative, may be suggested. This is based on the correlation of −0.91 between the cranial width and breadth-height index of 82 groups. The correction is obtained as above.

$$r\frac{\sigma x}{\sigma y}$$

$$-0.91\,\frac{4.44\,x}{5.06\,y}$$

$$x = -0.80\,y.$$

In the following are given the coefficient of correlation and the corrections for each diameter, together with examples to illustrate the use of the correction.

Coefficient of correlation, basion-nasion and cranial length, 76 groups	+0.65
Correction	+1.49
Coefficient of correlation, cranial width and cranial module index, 90 groups	+0.71
Correction	+0.99
Coefficient of correlation, cranial width and breadth-height index, 82 groups	−0.91
Correction	−0.80

	LENGTH	WIDTH	HEIGHT	BASION-NASION
Patagonians del Rio, undeformed	188.31	138.23	142.32	104.00
Patagonians del Rio, deformed	176.69	142.65	144.44	105.23
Correlated groups[1]	181.85	139.87		101.06

[1] These include the various racial groups used in determining the inter-racial correlation.

1. Correction of maximum cranial length of deformed Patagonians del Rio

Basion-nasion, deformed Patagonians	105.23
Basion-nasion, correlated groups	101.06
	4.17
	4.17
Correction	×1.49
	6.21
Maximum cranial length, correlated groups	181.85
	6.21
Corrected maximum cranial length of deformed Patagonians	188.06

2. Correction of maximum cranial width of deformed Patagonians del Rio

Cranial module index, deformed Patagonians	82.20
Cranial module index, correlated groups	83.74
	−1.54
	−1.54
Correction	× 0.99
	−1.52

Maximum cranial width, correlated groups	139.87
	−1.52
Corrected maximum cranial width of deformed Patagonians	138.35

3. Correction of cranial height of deformed Patagonians del Rio

Corrected cranial width, deformed Patagonians	138.35
Corrected cranial width, correlated groups	139.89
	−1.54
	−1.54
Correction	×−0.80
	+1.23
Breadth-height index, correlated groups	96.88
	1.23
Corrected breadth-height index, deformed Patagonians	98.11
	98.11
Corrected cranial width, deformed Patagonians	×138.35
Corrected cranial height, deformed Patagonians	135.74